Prove For Every Integer N > 7 That There Exist Positive Integers A And B Such That N = 2a + 3b.

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Introduction

The problem of expressing integers as combinations of smaller integers has fascinated mathematicians for centuries. One classic example is the study of representing numbers as linear combinations of two fixed positive integers, such as 2 and 3. This problem is rooted in the broader context of linear Diophantine equations, which are equations seeking integer solutions to polynomial equations, particularly linear ones.

Specifically, our goal is to prove that for every integer $N > 7$, there exist positive integers a and b such that

$$
N = 2a + 3b.
$$

This statement touches on the foundational concepts of number theory, including the Frobenius coin problem, which deals with the largest integer that cannot be expressed as a linear combination of two positive integers with certain properties.

Context and Significance

Understanding how integers can be expressed as sums of multiples of fixed integers is not only a theoretical pursuit but also has practical implications. For example, in coinage systems, determining which amounts can be formed using certain denominations is fundamental. The problem at hand can be viewed as a simplified version of this coin problem with denominations 2 and 3.

Historically, the Frobenius coin problem states that for two positive coprime integers m and n, the largest integer that cannot be expressed as a non-negative combination of m and n is $mn - m - n$. Since 2 and 3 are coprime, the largest integer that cannot be expressed as a non-negative integer combination of 2 and 3 is $2 \times 3 - 2 - 3 = 1$. All integers greater than 1 can be expressed as such combinations, but the classical Frobenius problem considers non-negative solutions, whereas our problem emphasizes positive integers a and b, adding a layer of nuance.

Clarifying the Problem Statement

Before delving into the proof, it is important to clarify the difference between non-negative and positive solutions:

- Non-negative solutions: $a, b \geq 0$
- Positive solutions: $a, b \geq 1$

Our task is to prove that for all integers $N > 7$, positive integers \(

a and b exist such that

$$
N = 2a + 3b,
$$

which is slightly stricter than the standard Frobenius problem since it excludes solutions where $a = 0$ or $b = 0$.

Strategy for the Proof

To establish the statement, we'll employ mathematical induction and constructive methods, including:

1. Verifying the base cases for initial values of N
2. Showing that if the statement holds for certain integers, it holds for larger integers
3. Demonstrating that all integers greater than 7 can be built up using the base cases and the building blocks 2 and 3

Step 1: Base Cases

Let's check the smallest integers greater than 7 and see if they can be expressed as $2a + 3b$ with $a, b \geq 1$:

- $N = 8$:

Try $b = 1$:

$$
8 = 2a + 3(1) \Rightarrow 8 = 2a + 3 \Rightarrow 2a = 5 \Rightarrow a = 2.5,
$$

not an integer.

Try $b = 2$:

$$
8 = 2a + 6 \Rightarrow 2a = 2 \Rightarrow a = 1,
$$

which is positive. So, $N=8$ can be expressed as $2 \times 1 + 3 \times 2$.

- $N=9$:

Try $b=1$:

$$
9=2a+3(1)=2a+3 \Rightarrow 2a=6 \Rightarrow a=3,
$$

positive. So, $9 = 2 \times 3 + 3 \times 1$.

- $N=10$:

Try $b=2$:

$$

$$
10=2a+6 \Rightarrow 2a=4 \Rightarrow a=2,
$$

positive. So, $10=2 \times 2 + 3 \times 2$.

- $N=11$:

Try $b=3$:

$$
11=2a+9 \Rightarrow 2a=2 \Rightarrow a=1,
$$

positive. So, $11=2 \times 1 + 3 \times 3$.

- $N=12$:

Try $b=4$:

$$
12=2a+12 \Rightarrow 2a=0 \Rightarrow a=0,
$$

but $a=0$ is not positive, so discard.

Try $b=3$:

$$
12=2a+9 \Rightarrow 2a=3 \Rightarrow a=1.5,
$$

discard.

Try $b=2$:

$$
12=2a+6 \Rightarrow 2a=6 \Rightarrow a=3,
$$

positive, so $12 = 2 \times 3 + 3 \times 2$.

- $N=13$:

Try $b=4$:

$$
13=2a+12 \Rightarrow 2a=1 \Rightarrow a=0.5,
$$

discard.

Try $b=3$:

$$
13=2a+9 \Rightarrow 2a=4 \Rightarrow a=2,
$$

positive. So, $13=2 \times 2 + 3 \times 3$.

- $N=14$:

Try $b=4$:

$$
14=2a+12 \Rightarrow 2a=2 \Rightarrow a=1,
$$

positive. So, $14=2 \times 1 + 3 \times 4$.

- $N=15$:

Try $b=5$:

$$
15=2a+15 \Rightarrow 2a=0 \Rightarrow a=0,
$$

discard.

$b=4$:

$$
15=2a+12 \Rightarrow 2a=3, \quad a=1.5,
$$

discard.

$b=3$:

$$
15=2a+9 \Rightarrow 2a=6, \quad a=3,
$$

positive. So, $15=2 \times 3 + 3 \times 3$.

- $N=16$:

$b=5$:

$$
16=2a+15 \Rightarrow 2a=1, \quad a=0.5,
$$

discard.

$b=4$:

$$
16=2a+12 \Rightarrow 2a=4, \quad a=2,
$$

positive. So, $16=2 \times 2 + 3 \times 4$.

From these base cases, we observe that for all N from 8 to 16, there exist positive integers a and b such that $N=2a+3b$.

Step 2: Inductive Hypothesis

Assume that for some integer $N \geq 8$, the statement holds: there exist positive integers a and b such that

$$
N = 2a + 3b.
$$

Our goal is to prove that the statement also holds for $N+1$, i.e.,

$$
N+1 = 2a' + 3b'
$$

for some positive integers a', b'.

Step 3: Inductive Step

To move from N to $N+1$, note that:

$$
N+1 = N + 1.
$$

Since $N = 2a + 3b$, adding 1 to N could be achieved by adjusting a and b. However, because a and b are positive integers, we need to find a way to express $N+1$ as $2a' + 3b'$ with $a', b' \geq 1$.

Key Idea:

- If $N-1$ can be expressed as $2a'' + 3b''$, then

$$
N = (N-1
$$

Frequently Asked Questions

How can we prove that for every integer N > 7, there exist positive integers A and B such that N = 2A + 3B?

We can prove this by using mathematical induction or by demonstrating that for N > 7, N can be expressed as a linear combination of 2 and 3 with positive coefficients. Since 2 and 3 are coprime, their linear combinations cover all sufficiently large integers, and specifically, N > 7 can be represented as such with positive A and B.

Why is the coprimality of 2 and 3 important in proving the existence of positive integers A and B for N > 7?

Because 2 and 3 are coprime, their linear combinations can generate all integers greater than a certain minimum. This property ensures that for sufficiently large N, there exist positive integers A and B satisfying N = 2A

+ 3B, especially for N > 7.

Can you provide an example demonstrating that N = 10 can be expressed as 2A + 3B with positive A and B?

Yes. For N = 10, one possible solution is A = 2 and B = 2 because 10 = 2×2 + 3×2 = 4 + 6 = 10. Both A and B are positive integers, confirming the statement.

Is the statement true for all integers N > 7, and how do we handle the boundary cases?

Yes, the statement is true for all integers N > 7. For boundary cases (like N = 8 or 9), explicit solutions can be found: for example, 8 = 2×4 + 3×0 is invalid since B must be positive, so 8 = 2×1 + 3×2 = 2 + 6 = 8 with A=1, B=2; similarly for N=9, 9=2×0 + 3×3, but since A must be positive, we find alternative solutions such as 9=2×3 + 3×1.

What is the significance of this proof in the context of number theory or integer representations?

This proof illustrates the concept of representing integers as linear combinations of coprime positive integers, highlighting key ideas in number theory related to the Frobenius coin problem and the ability to express large enough integers as combinations of given coprime numbers with positive coefficients.

Additional Resources

Mathematical Proof: Demonstrating that every integer $N > 7$ can be expressed as $N = 2a + 3b$

Introduction

Mathematics often presents us with intriguing questions that challenge our understanding of numbers and their relationships. One such question is whether every sufficiently large integer can be expressed as a linear combination of small positive integers. Specifically, this article explores the claim:

For every integer $N > 7$, there exist positive integers a and b such that

$$N = 2a + 3b$$

This statement is not just a theoretical curiosity; it embodies fundamental principles related to number theory, linear combinations, and the Frobenius coin problem. Our goal is to prove this assertion rigorously, employing comprehensive reasoning and step-by-step analysis, much like a detailed product review or expert feature that dissects complex concepts into digestible insights.

Understanding the Statement

Before diving into the proof, it is essential to understand the core elements of the statement:

- Positive integers a, b: Both a and b are integers greater than zero.
- Linear combination of 2 and 3: The expression $2a + 3b$ is a linear combination with positive coefficients.
- Threshold $N > 7$: The statement applies to all integers greater than 7, emphasizing that smaller values may not necessarily fit the pattern, but beyond this point, the pattern holds universally.

The challenge lies in demonstrating that for any $N > 7$, we can find suitable positive a, b satisfying the equation.

Why Is This Important?

Proving this property has implications in various domains:

- Number theory: It relates to the Frobenius coin problem, which deals with representations of numbers as combinations of coprime integers.
- Algorithm design: Understanding such representations aids in resource allocation and optimization.
- Mathematical completeness: It confirms the capacity to generate large numbers from small building blocks, a key idea in additive number theory.

Approach to the Proof

To establish the claim, we'll employ a constructive proof approach, which involves explicitly demonstrating how to find a and b for any $N > 7$. The core methods include:

- Base cases: Verifying the statement for initial values above 7.
- Inductive reasoning: Showing that if the statement holds for a certain N, it also holds for $N + 1$.
- Decomposition: Breaking down larger numbers into sums involving smaller numbers that are known to be representable.

Our proof will also leverage the properties of the numbers 2 and 3, which are coprime, meaning their greatest common divisor (gcd) is 1. This coprimality is crucial because it guarantees that sufficiently large integers can be expressed as linear combinations of 2 and 3 with positive coefficients.

The Foundation: Coprimality of 2 and 3

Key property: Since 2 and 3 are coprime ($\gcd(2, 3) = 1$), every integer N greater than or equal to \) a certain number can be written as a linear combination $N = 2a + 3b$ with integer coefficients.

Frobenius Number: The largest integer that cannot be expressed as a non-negative linear combination of 2 and 3 is known as the Frobenius number for these two numbers, which is:

$$
g(2, 3) = 2 \times 3 - 2 - 3 = 6 - 2 - 3 = 1
$$

This means:

- All integers greater than 1 can be expressed as a non-negative linear combination of 2 and 3.
- Specifically, for non-negative integers $a, b \geq 0$, every $N \geq 2$ can be written as $N = 2a + 3b$.

However, our problem requires positive integers a, b, which means $a, b \geq 1$. This subtlety is important because the standard Frobenius number deals with non-negative coefficients, and our problem restricts to positive coefficients.

Establishing the Core: Representations with Positive a, b

Let's analyze the initial small values to understand the pattern:

N	Possible representations $N = 2a + 3b$ with $a, b \geq 1$
4	$2 \times 2 = 4$ (but $a=2, b=0$), b=0 not allowed; no b=1: $2a + 3(1)= 2a + 3$. For $a=1$, $2 + 3=5 \neq 4$; for $a=2$, $4+3=7 \neq 4$. No solution.
5	$2a + 3b=5$. Try $b=1$: $2a + 3=5 \Rightarrow 2a=2 \Rightarrow a=1$. Valid: $a=1, b=1$.
6	$2a + 3b=6$. $b=1 \Rightarrow 2a+3=6 \Rightarrow 2a=3$, no. $b=2 \Rightarrow 2a+6=6 \Rightarrow 2a=0 \Rightarrow a=0$, but $a=0$ not positive. $b=0 \Rightarrow 2a=6 \Rightarrow a=3$, valid but $b=0$, not positive. So no $a,b \geq 1$.
7	$2a+3b=7$. $b=1 \Rightarrow 2a+3=7 \Rightarrow 2a=4 \Rightarrow a=2$, valid $a=2, b=1$.

From this, we observe:

- For $N=5$, representation exists with positive integers.
- For $N=7$, representation exists.
- For $N=6$, no, since a or b would be zero or negative.

This suggests that for larger N, especially beyond 7, representations with positive a, b are more readily obtainable.

Formal Proof: Every $N > 7$ can be expressed as $N=2a+3b$ with $a, b \in \mathbb{Z}^+$

Step 1: Base cases

Check explicitly for $N=8, 9, 10, 11, 12$:

- $N=8$:

Try $b=1 \Rightarrow 2a + 3=8 \Rightarrow 2a=5$, no.

$b=2 \Rightarrow 2a+6=8 \Rightarrow 2a=2 \Rightarrow a=1$. Valid: $a=1, b=2$.

- $N=9$:

$b=1 \Rightarrow 2a+3=9 \Rightarrow 2a=6 \Rightarrow a=3$. Valid.

- $N=10$:

$b=2 \Rightarrow 2a+6=10 \Rightarrow 2a=4 \Rightarrow a=2$. Valid.

- $N=11$:

$b=1 \Rightarrow 2a+3=11 \Rightarrow 2a=8 \Rightarrow a=4$. Valid.

- $N=12$:

$b=2 \Rightarrow 2a+6=12 \Rightarrow 2a=6 \Rightarrow a=3$. Valid.

Thus, for all N from 8 to 12, explicit positive solutions exist.

Step 2: Inductive Step

Assuming that for some $N \geq 12$, there exist positive integers a', b' such that:

$$
N = 2a' + 3b'
$$

Our goal is to show that $N+1$ can also be expressed as $2a'' + 3b''$ with positive integers a'', b''.

Observe that:

$$
N + 1 = (2a' + 3b') + 1
$$

Note that:

$$
N + 1 = 2a' + 3b' + 1
$$

But since $2a' + 3b' = N$, and we want $N+1 = 2a'' + 3b''$ with positive a'', b'', consider the following:

- If we can find a way to increase a' or b' by 1 or adjust their values to reach $N+1$, maintaining positivity.

Alternatively, note that:

$$
N+1 = 2(a' +

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