Show That The Union Of A Nonempty Directed Family Of Subgroups Of A Group G Is A Subgroup Of G?

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Understanding how subgroups combine within a larger group is a fundamental aspect of group theory, a core area of abstract algebra. One intriguing question is under what conditions the union of multiple subgroups itself forms a subgroup. Specifically, if we have a nonempty family of subgroups of a group G, and this family is directed, then the union of these subgroups is guaranteed to be a subgroup of G. This result is not only elegant but also essential in understanding structures like directed systems and direct limits in algebra.

In this article, we will explore this concept thoroughly. We will define what a directed family of subgroups is, and then carefully demonstrate why their union under these conditions is a subgroup. Along the way, examples and key properties will be discussed to solidify understanding.

Preliminaries and Definitions

Before diving into the main proof, it’s important to establish clear definitions and notations.

Subgroups of a Group G

A subset $H \subseteq G$ is called a subgroup of G if it satisfies the following conditions:

- Closure: For all $a, b \in H$, the product $ab \in H$.
- Identity: The identity element $e \in H$.
- Inverses: For all $a \in H$, the inverse $a^{-1} \in H$.

Family of Subgroups and Their Union

Suppose $\mathcal{F} = \{ H_\alpha \mid \alpha \in A \}$ is a family of subgroups of G, indexed by some set A. The union of this family is:

$$
\bigcup_{\alpha \in A} H_\alpha = \{ x \in G \mid x \in H_\alpha \text{ for some } \alpha \in A \}.
$$

A natural question arises: Is this union always a subgroup? The answer, in general, is no. The union of subgroups need not be a subgroup unless additional conditions are satisfied.

Directed Families of Subgroups

The key condition that ensures the union of subgroups is itself a subgroup is the directedness of the family.

Definition of a Directed Family

A family $\mathcal{F} = \{ H_\alpha \}_{\alpha \in A}$ of subgroups of G is called directed if:

- $\mathcal{F}$ is nonempty; that is, there exists at least one $H_\alpha \in \mathcal{F}$.
- For any two subgroups $H_\alpha, H_\beta \in \mathcal{F}$, there exists another subgroup $H_\gamma \in \mathcal{F}$ such that:

$$
H_\alpha \subseteq H_\gamma \quad \text{and} \quad H_\beta \subseteq H_\gamma.
$$

In other words, given any two subgroups in the family, there is a third subgroup that contains both.

This directedness condition ensures the family is "upward-directed" or "coherent" in the sense of inclusion, which is crucial for the union to inherit subgroup properties.

The Main Theorem and Its Proof

Theorem:
Let G be a group, and $\mathcal{F} = \{ H_\alpha \}_{\alpha \in A}$ be a nonempty, directed family of subgroups of G. Then, the union $H = \bigcup_{\alpha \in A} H_\alpha$ is a subgroup of G.

Step 1: Show that H contains the identity element e.

Since each H_α is a subgroup, it contains e. Given that $\mathcal{F}$ is nonempty, pick any $H_{\alpha_0} \in \mathcal{F}$. Then:

$$
e \in H_{\alpha_0} \subseteq H,
$$

so the union H contains e.

Step 2: Show that H is closed under inverses.

Take any element $x \in H$. By the definition of union, there exists some $H_\alpha \in \mathcal{F}$ such that:

$$
x \in H_\alpha.
$$

Since H_α is a subgroup, it contains the inverse x^{-1}. Because $x^{-1} \in H_\alpha$, it follows that:

$$
x^{-1} \in H.
$$

Thus, H is closed under inverses.

Step 3: Show that H is closed under multiplication.

This is the critical part. Take any $x, y \in H$. Then:

- There exist $H_\alpha, H_\beta \in \mathcal{F}$ such that
$$
x \in H_\alpha, \quad y \in H_\beta.
$$

Because $\mathcal{F}$ is directed, there exists some $H_\gamma \in \mathcal{F}$ such that:

$$
H_\alpha \subseteq H_\gamma \quad \text{and} \quad H_\beta \subseteq H_\gamma.
$$

Since both $x, y \in H_\gamma$ (by inclusion), and H_γ is a subgroup, it contains the product:

$$
xy \in H_\gamma.
$$

Therefore,

$$
xy \in H,
$$

since $H_\gamma \subseteq H$. This confirms that H is closed under multiplication.

Conclusion: H is a Subgroup of G

Having verified the three subgroup criteria—containing the identity, closed under inverses, and closed under multiplication—it's clear that:

$$
H = \bigcup_{\alpha \in A} H_\alpha
$$

is a subgroup of G.

Implications and Applications

This result has several important implications in algebra and related fields.

Construction of Direct Limits

In category theory and algebra, the concept of a direct limit involves taking a directed system of groups (or modules, rings, etc.) and forming their union in a way that preserves structure. The theorem ensures that the union of a directed family of subgroups is itself a subgroup, serving as a building block for direct limits.

Building Larger Subgroups from Smaller Ones

Often, complex groups are constructed as unions of smaller subgroups, especially in infinite groups. The theorem assures us that when these smaller subgroups are arranged in a directed manner, their union can be treated as a new subgroup, facilitating analysis and further constructions.

Understanding the Structure of Infinite Groups

Many infinite groups are studied via chains or directed systems of subgroups. Recognizing when unions are subgroups simplifies the analysis of their structure, particularly in the context of solvable groups, nilpotent groups, and other classes where ascending chains are common.

Examples to Illustrate the Theorem

Example 1:
Let $G = \mathbb{Q}$ (the additive group of rational numbers). Define the family:

$$
H_n = \left\{ \frac{m}{2^n} \mid m \in \mathbb{Z} \right\}
$$

for each positive integer n. Each H_n is a subgroup of $\mathbb{Q}$, and the family is directed because:

$$
H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots
$$

The union:

$$
H = \bigcup_{n=1}^\infty H_n
$$

is the set of all rational numbers with denominators powers of 2, which is a subgroup of $\mathbb{Q}$.

Example 2:
Suppose G is a group, and $\{ H_\alpha \}$ is a family of subgroups such that for any two H_α, H_β, there exists H_γ containing both. Their union is then a subgroup. Conversely, if the family is not directed, the union may fail to be a subgroup (for example, the union of two subgroups that are not contained within a common larger

subgroup may not be a subgroup).

Summary

To summarize, the key points are:

- The union of a nonempty family of subgroups of a group G is not necessarily a subgroup.
- When the family is directed, meaning for any two subgroups there exists a larger subgroup containing both, the union is a subgroup.
- The proof hinges

Frequently Asked Questions

How can we prove that the union of a nonempty directed family of subgroups of a group G is itself a subgroup of G?

To prove this, assume $\{H_i\}$ is a nonempty directed family of subgroups of G. Since the family is directed, for any H_i and H_j, there exists H_k such that $H_i, H_j \subseteq H_k$. The union $H = \bigcup H_i$ is nonempty and closed under the group operation and inverses because for any elements a, b in H, they belong to some H_m and H_n, respectively; then, due to the directedness, there exists H_k containing both, ensuring that the product $a \cdot b$ and the inverse a^{-1} are also in H_k, hence in H. Therefore, H is a subgroup of G.

What is the significance of the family being directed in establishing that the union is a subgroup?

The directedness ensures that for any two subgroups in the family, there exists a larger subgroup containing both. This property guarantees that the union is closed under group operations and inverses, which is essential for the union to be a subgroup. Without directedness, the union may fail to be closed under these operations.

Can the union of a nonempty family of subgroups be a subgroup if the family is not directed?

Not necessarily. If the family is not directed, the union may fail to be closed under the group operation or inverses. For example, the union might contain elements whose product or inverse is not in the union, preventing it from being a subgroup.

Is the union of an arbitrary collection of subgroups always a subgroup?

No. The union of an arbitrary collection of subgroups is not necessarily a subgroup. It is only guaranteed to be a subgroup if the family is nonempty and directed. Without these conditions, the union may lack closure under group operations.

How does the concept of directed families relate to the structure of groups and their subgroups?

Directed families of subgroups are important in the study of group filters, chains, and inductive limits. They facilitate constructing larger subgroups from smaller ones while maintaining certain properties, and are key in understanding subgroup hierarchies and the formation of subgroups via unions.

Can you give an example of a directed family of subgroups whose union is a subgroup of G?

Yes. Consider G = $\mathbb{Z}$ (the integers), and the family $\{2^n\mathbb{Z} \mid n \geq 0\}$. This family is directed since for any two subgroups $2^m\mathbb{Z}$ and $2^n\mathbb{Z}$, there exists $2^{\max(m,n)}\mathbb{Z}$ containing both. Their union is the set of all integers divisible by some power of 2, which is the union over all n of $2^n\mathbb{Z}$, and this union is itself a subgroup of $\mathbb{Z}$.

Additional Resources

Show That The Union Of A Nonempty Directed Family Of Subgroups Of A Group G Is A Subgroup Of G?

Understanding the structure of groups and their subgroups is fundamental in abstract algebra. One intriguing aspect concerns how collections of subgroups relate to the larger group they inhabit. Specifically, mathematicians are often interested in whether the union of certain families of subgroups retains the subgroup properties themselves. This question is not only intellectually stimulating but also foundational for more advanced topics like the construction of direct limits, the study of chains of subgroups, and the classification of algebraic structures.

In this article, we explore the assertion: Given a nonempty directed family of subgroups of a group G, their union forms a subgroup of G. We will unpack this statement through a careful, step-by-step analysis, providing the necessary definitions, theorems, and proof strategies to deepen your understanding of this fundamental concept in group theory.

The Foundations: Subgroups, Unions, and Directed Families

Before diving into the proof, it's essential to clarify the key concepts involved.

What Is a Subgroup?

A subgroup H of a group G is a subset of G that itself forms a group under the operation inherited from G. Formally, $H \subseteq G$ is a subgroup if:

- The identity element e of G is in H.
- For every h, h' in H, the product $h \cdot h'$ is also in H.
- For every h in H, the inverse h^{-1} is also in H.

These properties ensure that the subgroup maintains the algebraic structure within the larger group.

Union of Subgroups

Given a family of subgroups $\{H_\alpha \mid \alpha \in A\}$ of G, their union is simply:

$$ \bigcup_{\alpha \in A} H_\alpha = \{ g \in G \mid g \in H_\alpha \text{ for some } \alpha \in A \} $$

The question is: does this union necessarily form a subgroup? Generally, the union of two subgroups is not guaranteed to be a subgroup unless certain conditions are met.

The Notion of a Directed Family

A family of subgroups $\{H_\alpha\}$ is called directed (or a directed system) if for any two subgroups H_α and H_β in the family, there exists another subgroup H_γ that contains both H_α and H_β:

$$ \forall \alpha, \beta \in A, \quad \exists \gamma \in A \text{ such that } H_\alpha \subseteq H_\gamma \text{ and } H_\beta \subseteq H_\gamma $$

Intuitively, this means the subgroups are "coherently increasing" or "nested" in a way that allows their union to be well-behaved.

The Core Theorem: The Union of a Nonempty Directed Family of Subgroups Is a Subgroup

The central claim can now be formalized:

Theorem:
Let $\{H_\alpha \mid \alpha \in A\}$ be a nonempty directed family of subgroups of G. Then their union:

$$ H = \bigcup_{\alpha \in A} H_\alpha $$

is itself a subgroup of G.

This theorem highlights the significance of the directedness condition: it ensures that the union preserves the subgroup structure. The proof hinges on verifying the subgroup criteria—closure under the group operation and taking inverses—within this union.

Detailed Proof and Explanation

Let's delve into the proof, step-by-step, emphasizing the intuition and logical flow.

Step 1: Nonemptiness

Since the family is nonempty, there exists at least one subgroup H_α in the family. Because subgroups contain the identity element e of G, it follows that:

$$ e \in H_\alpha \subseteq H $$

Thus, the union H contains the identity element, satisfying the first subgroup criterion.

Step 2: Closure Under Inverses

Suppose $g \in H$. By definition of the union, g belongs to some H_α:

$$ g \in H_\alpha $$

Since H_α is a subgroup, it is closed under inverses:

$$ g^{-1} \in H_\alpha \subseteq H $$

Therefore, the inverse of any element in H also belongs to H, satisfying the second subgroup criterion.

Step 3: Closure Under the Group Operation

The critical part of the proof involves showing that the product of any two elements in H remains in H.

Let $g, h \in H$. By the union's definition, there exist $\alpha, \beta \in A$ such that:

$$ g \in H_\alpha, \quad h \in H_\beta $$

Now, the key is to determine whether $g \cdot h \in H$. Since the family $\{H_\alpha\}$ is directed, for any α and β, there exists $\gamma \in A$ such that:

$$ H_\alpha \subseteq H_\gamma \quad \text{and} \quad H_\beta \subseteq H_\gamma $$

Because H_γ is a subgroup, it contains both g and h:

$$ g, h \in H_\gamma $$

and being a subgroup, H_γ is closed under multiplication:

$$ g \cdot h \in H_\gamma \subseteq H $$

Thus, $g \cdot h$ belongs to H, satisfying the closure property.

Summarizing the Proof

The proof leverages the properties of the directed family of subgroups to ensure that the union is closed under the group operation and inverses. The critical step is the directedness condition, which guarantees that for any two elements originating from different subgroups, there's a larger subgroup containing both, thereby controlling the union's behavior.

Key Takeaway:

- The union of a nonempty directed family of subgroups is always a subgroup.
- Without the directedness condition, the union may fail to be closed under the group operation, and hence may not be a subgroup.

Why Is The Directedness Condition Necessary?

To appreciate the importance of the directedness condition, consider what could go wrong if it is absent.

Suppose $\{H_\alpha\}$ is a family of subgroups where, for some α and β, no larger subgroup contains both H_α and H_β. Then, elements $g \in H_\alpha$ and $h \in H_\beta$ might not be contained together in any single H_γ, and their product $g \cdot h$ could fall outside the union. This would violate closure under the group operation.

Example:

Imagine $G = \mathbb{Z}$ (the integers), with subgroups:

- $H_1 = 2\mathbb{Z}$ (even integers)
- $H_2 = 3\mathbb{Z}$ (multiples of 3)

Their union is the set of integers divisible by 2 or 3, which is not a subgroup because the sum of an even integer and a multiple of 3 need not be divisible by 2 or 3. The union $H_1 \cup H_2$ isn't closed under addition unless one of the subgroups contains the other, which is only true if one is contained within the other (i.e., the family is directed).

Applications and Broader Implications

Understanding how unions of subgroups behave is more than an academic exercise; it underpins several constructions and theorems in group theory and beyond.

Constructing Larger Subgroups

The concept allows mathematicians to build larger subgroups from chains or directed systems of smaller subgroups. For example, in the theory of direct limits or inductive systems, the union of an increasing chain of subgroups forms the union, which is then itself a subgroup under the directedness condition.

The Role in Topology and Algebraic Geometry

Similar concepts appear in topology with directed systems of spaces and in algebraic geometry with directed systems of schemes or modules. The underlying idea is that controlled unions preserve structure when certain conditions (like directedness) are met.

Subgroup Chains and Classification

Classifying groups often involves analyzing chains of subgroups and their unions. The theorem assures that such unions, when formed from directed families, will always yield subgroups, facilitating the study of the group's internal hierarchy.

Concluding Remarks

The proof that the union of a nonempty directed family of subgroups of a group G is itself a subgroup is a cornerstone in understanding the layered structure of groups. It highlights how the property of directedness ensures the preservation of algebraic structure under unions, enabling a systematic approach to building and analyzing complex groups from simpler components.

This foundational result exemplifies the elegance of abstract algebra: a simple condition (directedness) guarantees a powerful structural property. Recognizing and applying such theorems is vital for advancing in the field, whether in pure mathematical research or in applications spanning physics, computer science, and beyond.

By appreciating the nuances of subgroup unions and the importance of the directedness condition, mathematicians can leverage these insights to unlock deeper understandings of algebraic systems, their classifications, and their interrelations.

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