Solve The Following Differential Equations. (a) Y'' + 4y = X Sin 2x. (b) Y' = 1+3y (c) Y'' - 6y = 0.

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Introduction to Differential Equations

Differential equations are fundamental mathematical tools used to describe various phenomena in physics, engineering, biology, economics, and many other fields. They relate a function to its derivatives, enabling us to model dynamic systems and predict future behavior based on current conditions. Solving differential equations involves finding the original function that satisfies the given relation.

In this article, we will explore three specific differential equations:

- (a) $Y'' + 4y = x \sin 2x$
- (b) $Y' = 1 + 3y$
- (c) $Y'' - 6y = 0$

We will dissect each problem, understand the type of differential equation it represents, and walk through step-by-step methods to find their solutions.

Solving Differential Equation (a): $Y'' + 4y = x \sin 2x$

1. Recognizing the Type: Nonhomogeneous Second-Order Differential Equation

The equation

$$
Y'' + 4y = x \sin 2x
$$

is a second-order linear nonhomogeneous differential equation. The homogeneous part is:

$$
Y'' + 4y = 0
$$

and the nonhomogeneous part (forcing function) is:

$$
f(x) = x \sin 2x
$$

2. Solving the Homogeneous Equation

First, find the complementary solution y_c:

$$
Y'' + 4y = 0
$$

The characteristic equation:

$$
r^2 + 4 = 0
$$

which yields roots:

$$
r = \pm 2i
$$

The general solution to the homogeneous equation:

$$
y_c = C_1 \cos 2x + C_2 \sin 2x
$$

where C_1 and C_2 are arbitrary constants.

3. Finding a Particular Solution

Since the right-hand side is $x \sin 2x$, which involves a product of a polynomial and a trigonometric function, we can use the Method of Undetermined Coefficients or Variation of Parameters.

Note: Because $\sin 2x$ and $\cos 2x$ are solutions of the homogeneous equation, the method of undetermined coefficients may need modification (e.g., multiplying by x).

Preferred method here: Variation of Parameters.

4. Applying Variation of Parameters

The general form for the particular solution:

$$
y_p = u_1(x) \cos 2x + u_2(x) \sin 2x
$$

where \(u_1' \) and \(u_2' \) satisfy:

\[
\begin{cases}
u_1' \cos 2x + u_2' \sin 2x = 0 \\
-2 u_1' \sin 2x + 2 u_2' \cos 2x = \frac{x \sin 2x}{1}
\end{cases}
\]

but since the right side is complicated, the standard variation of parameters formulas for \(u_1' \) and \(u_2' \) are:

\[
u_1' = - \frac{f(x) \sin 2x}{W}
\]
\[
u_2' = \frac{f(x) \cos 2x}{W}
\]

where \(W \) is the Wronskian:

\[
W = \det \begin{bmatrix}
\cos 2x & \sin 2x \\
-2 \sin 2x & 2 \cos 2x
\end{bmatrix} = 2
\]

Calculating \(u_1' \) and \(u_2' \):

\[

$$
u_1' = - \frac{x \sin 2x \times \sin 2x}{2} = - \frac{x \sin^2 2x}{2}
$$

$$
u_2' = \frac{x \sin 2x \times \cos 2x}{2}
$$

Integrate to find u_1 and u_2:

$$
u_1(x) = - \frac{1}{2} \int x \sin^2 2x \, dx
$$

$$
u_2(x) = \frac{1}{2} \int x \sin 2x \cos 2x \, dx
$$

These integrals can be computed using standard calculus techniques, such as substitution and integration by parts.

5. Final Solution

The general solution:

$$
Y = y_c + y_p = C_1 \cos 2x + C_2 \sin 2x + y_p
$$

where y_p is obtained from the integrals above.

Solving Differential Equation (b): $Y' = 1 + 3y$

1. Recognizing the Type: First-Order Linear Differential Equation

The equation

$$
Y' = 1 + 3y
$$

can be rewritten as:

$$
\frac{dy}{dx} - 3y = 1
$$

which is a first-order linear differential equation in standard form.

2. Standard Form and Integrating Factor

The standard form:

$$
\frac{dy}{dx} + P(x) y = Q(x)
$$

with

$$
P(x) = -3, \quad Q(x) = 1
$$

The integrating factor:

\[
\mu(x) = e^{\int P(x) dx} = e^{\int -3 dx} = e^{-3x}
\]

3. Solving the Equation

Multiply both sides by the integrating factor:

\[
e^{-3x} \frac{dy}{dx} - 3 e^{-3x} y = e^{-3x}
\]

which simplifies to:

\[
\frac{d}{dx} [y e^{-3x}] = e^{-3x}
\]

Integrate both sides:

\[
y e^{-3x} = \int e^{-3x} dx + C
\]

\[
\int e^{-3x} dx = -\frac{1}{3} e^{-3x} + C'
\]

So,

\[

$$
y e^{-3x} = -\frac{1}{3} e^{-3x} + C
$$

Multiply through by e^{3x}:

$$
y = -\frac{1}{3} + C e^{3x}
$$

where C is an arbitrary constant.

4. Final Solution

$$
\boxed{
y(x) = -\frac{1}{3} + C e^{3x}
}
$$

This expression provides the general solution to the differential equation.

Solving Differential Equation (c): $Y'' - 6y = 0$

1. Recognizing the Type: Homogeneous Second-Order Differential Equation

This is a homogeneous linear second-order differential equation with constant coefficients:

$$
Y'' - 6y = 0
$$

\]

2. Characteristic Equation

Assuming solutions of the form \(y = e^{rx} \), substitute into the differential equation:

\[
r^2 e^{rx} - 6 e^{rx} = 0
\]

Divide both sides by \(e^{rx} \) (non-zero):

\[
r^2 - 6 = 0
\]

Solve for \(r \):

\[
r^2 = 6
\]
\[
r = \pm \sqrt{6}
\]

3. General Solution

The general solution:

\[
Y(x) = C_1 e^{\sqrt{6} x} + C_2 e^{-\sqrt{6} x}

\]

where C_1 and C_2 are arbitrary constants.

Summary and Key Takeaways

Types of Differential Equations Covered

Equation	Type	Solution Approach
$Y'' + 4y = x \sin 2x$	Second-order linear nonhomogeneous	Homogeneous + Variation of Parameters
$Y' = 1 + 3y$	First-order linear	Integrating Factor Method
$Y'' - 6y = 0$	Homogeneous second-order	Characteristic Equation

Common Methods Used

- Characteristic Equation: For constant coefficient linear differential equations.
- Method of Undetermined Coefficients: For certain nonhomogeneous equations with polynomial, exponential, or trigonometric right-hand sides.
- Variation of Parameters: When undetermined coefficients are cumbersome or inapplicable.
- Integrating Factor: For first-order linear differential equations.

Practical Applications of Differential Equations

Understanding how to solve these equations is vital in modeling real-world systems:

- Physics: Motion under forces, wave propagation

Frequently Asked Questions

How do you solve the differential equation Y'' + 4Y = X sin 2x?

First, find the complementary solution by solving Y'' + 4Y = 0, which has homogeneous solutions Y_c = C1 cos 2x + C2 sin 2x. Next, find a particular solution using the method of variation of parameters or undetermined coefficients. Since the RHS is X sin 2x, which resonates with the homogeneous solution, you may need to multiply the particular solution by X to account for duplication. The overall solution is Y = Y_c + Y_p.

What is the general solution for the differential equation Y' = 1 + 3Y?

This is a first-order linear differential equation. Rewrite as dY/dX - 3Y = 1. Integrate using integrating factor μ = e^{-3x}. The solution is Y = (1/3) + Ce^{3x}, where C is an arbitrary constant.

How do you solve the homogeneous differential equation Y'' - 6Y = 0?

Solve the characteristic equation r^2 - 6 = 0, giving r = ±□6. The general solution is Y = C1 e^{□6 x} + C2 e^{-□6 x}.

What methods are effective for solving second-order linear differential equations with constant coefficients?

The primary method is solving the characteristic equation to find the homogeneous solution. For nonhomogeneous equations, methods include undetermined coefficients or variation of parameters, depending on the form of the RHS.

How can resonance affect the particular solution when solving differential equations like Y'' + 4Y = X sin 2x?

Resonance occurs when the RHS frequency matches the natural frequency of the homogeneous solution. In such cases, the particular solution must be modified, typically multiplied by x (or higher powers of x) to account for the repeated roots, ensuring a valid particular solution is found.

Additional Resources

Solve The Following Differential Equations is a fundamental skill in advanced mathematics and engineering, providing essential tools to model real-world phenomena ranging from physics to economics. Mastering the methods to solve differential equations enables students and professionals to analyze systems' behavior over time, predict future states, and design solutions to complex problems. This article offers a comprehensive review of solving three specific types of differential equations: a second-order linear nonhomogeneous differential equation, a first-order nonlinear differential equation, and a second-order homogeneous differential equation. Each section will delve into the methods, solutions, and nuances involved, offering clarity and insight into their practical applications.

Part A: Solving Y'' + 4y = x sin 2x

This equation is a second-order linear nonhomogeneous differential equation with constant coefficients. It involves an unknown function y(x) and its second derivative Y''. The right-hand side (nonhomogeneous term) x sin 2x introduces complexities that require particular solution techniques.

Understanding the Equation

The differential equation:

$$ Y'' + 4y = x \sin 2x $$

has the homogeneous part:

$$ Y'' + 4y = 0 $$

and the nonhomogeneous term:

$$ x \sin 2x $$

which is a product of a polynomial and a trigonometric function.

Solving the Homogeneous Equation

The homogeneous equation:

$$ Y'' + 4y = 0 $$

has a characteristic equation:

$$ r^2 + 4 = 0 $$

$$ r^2 = -4 $$

$$ r = \pm 2i $$

The general solution to the homogeneous part:

$$y_h = C_1 \cos 2x + C_2 \sin 2x$$

where C_1, C_2 are arbitrary constants.

Finding the Particular Solution

Since the nonhomogeneous term involves $x \sin 2x$, which resonates with the solutions of the homogeneous equation (due to the $\sin 2x$ term), the method of undetermined coefficients must be adjusted accordingly.

- Method Used: Method of Variation of Parameters is suitable here because the nonhomogeneous term is a product involving polynomial and trigonometric functions, and the method handles such cases well.

Step-by-step process:

1. Determine the fundamental solutions:

$$y_1 = \cos 2x$$
$$y_2 = \sin 2x$$

2. Compute the Wronskian:

$$W = y_1 y_2' - y_1' y_2 = (\cos 2x)(2 \cos 2x) - (-2 \sin 2x)(\sin 2x)$$
$$W = 2 \cos^2 2x + 2 \sin^2 2x = 2 (\cos^2 2x + \sin^2 2x) = 2$$

3. Formulas for particular solution:

$$y_p = - y_1 \int \frac{y_2 \cdot g(x)}{W} dx + y_2 \int \frac{y_1 \cdot g(x)}{W} dx$$

where $g(x) = x \sin 2x$.

4. Calculations involve evaluating integrals:

$$
\int \frac{\sin 2x \cdot x \sin 2x}{2} dx, \quad \int \frac{\cos 2x \cdot x \sin 2x}{2} dx
$$

which are non-trivial and require integration by parts or advanced techniques.

Features:

- Pros: The method of variation of parameters is systematic and works for all types of functions.
- Cons: It can involve complicated integrations, especially with products of polynomials and trigonometric functions.

Final Solution

The complete solution:

$$ y(x) = y_h + y_p $$

where y_h is as above, and y_p is obtained from the integrals.

Summary: Solving this equation involves understanding the homogeneous solutions, recognizing the resonance due to the nonhomogeneous term, and applying variation of parameters carefully.

Part B: Solving Y' = 1 + 3y

This is a first-order linear differential equation with a nonlinear component due to the $3y$ term, which makes it a Bernoulli-type differential equation.

Understanding the Equation

The equation:

$$\frac{dy}{dx} = 1 + 3y$$

can be rearranged as:

$$\frac{dy}{dx} - 3y = 1$$

which is linear in form.

Method: Integrating Factor

Since the equation is linear:

$$y' - 3y = 1$$

it can be solved using the integrating factor:

$$\mu(x) = e^{\int -3 dx} = e^{-3x}$$

Multiplying through:

$$ e^{-3x} y' - 3 e^{-3x} y = e^{-3x} $$

which simplifies to:

$$ \frac{d}{dx} (e^{-3x} y) = e^{-3x} $$

Integrate both sides:

$$ e^{-3x} y = \int e^{-3x} dx + C $$

$$ e^{-3x} y = -\frac{1}{3} e^{-3x} + C $$

Multiply both sides by e^{3x}:

$$ y = -\frac{1}{3} + C e^{3x} $$

Features:

- Pros: Straightforward application of integrating factors makes this method quick.
- Cons: The solution includes an arbitrary constant, reflecting the generality.

Final solution:

$$ y(x) = -\frac{1}{3} + C e^{3x} $$

where C is an arbitrary constant determined by initial conditions.

Part C: Solving Y'' - 6y = 0

This is a second-order homogeneous linear differential equation with constant coefficients.

Understanding the Equation

The differential equation:

$$ Y'' - 6 y = 0 $$

has a characteristic equation:

$$ r^2 - 6 = 0 $$

$$ r^2 = 6 $$

$$ r = \pm \sqrt{6} $$

which are real and distinct roots.

Solution Method

The general solution to this homogeneous equation:

$$ y_h = C_1 e^{\sqrt{6} x} + C_2 e^{-\sqrt{6} x} $$

where C_1, C_2 are arbitrary constants.

Features:

- Pros: Straightforward root-based solution, no particular solution needed.
- Cons: The solution's behavior is exponential, which may not be suitable for certain boundary conditions unless bounded.

Overall Evaluation and Practical Considerations

Strengths of Differential Equation Solution Techniques:

- Systematic approaches: Methods like characteristic equations, undetermined coefficients, variation of parameters, and integrating factors provide clear pathways.
- Versatility: Applicable to a wide range of equations, from simple first-order to complex second-order nonhomogeneous equations.
- Predictive power: Solutions allow modeling and prediction of physical systems.

Limitations and Challenges:

- Complex integrations: Particularly in nonhomogeneous cases with products of polynomials and trig functions.
- Resonance cases: When the nonhomogeneous term is similar to homogeneous solutions, requiring modified methods.
- Initial conditions: Additional information is often necessary to determine arbitrary constants.

Key Features Summary:

Feature	Description	Pros	Cons
Homogeneous solution methods	Solve characteristic equations for roots	Straightforward for	

constant coefficients | Not applicable to nonhomogeneous equations |

| Undetermined coefficients | Guess particular solutions based on RHS form | Quick for specific RHS types | Fails with complex RHS, resonance cases |

| Variation of parameters | Systematic method for complex RHS | Works universally | Can involve complicated integrations |

| Integrating factor | Simplifies first-order linear ODEs | Very efficient for linear equations | Not applicable for nonlinear equations |

Conclusion

The process of solving the differential equations presented—ranging from second-order with nonhomogeneous terms to first-order nonlinear forms—highlights the richness and complexity of differential equation theory. Each method discussed has its domain of applicability and limitations, but together they form a comprehensive toolkit for tackling a wide array of problems. Whether employing characteristic equations for homogeneous cases, variation of parameters for nonhomogeneous equations, or integrating factors for first-order linear equations, understanding the underlying principles enables effective application. Mastery of these methods enhances one's ability to model real-world systems accurately, predict behaviors, and develop innovative solutions across scientific and engineering disciplines.

Final notes: Practical application of these methods often requires careful algebraic manipulation, integration, and sometimes numerical approximation when integrals become intractable analytically. Nonetheless, a solid grasp of these foundational techniques is invaluable for advanced

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