Solve The Initial Value Problem And Find Y(1): Yy' = Xe, Y(0) = 1 2 In(1 + E) In(1+E) O Ln 1 + E

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In this comprehensive guide, we will walk through the process of solving the initial value problem (IVP) involving the differential equation:

\[Y y' = Xe, \quad Y(0) = \frac{1}{2} \ln(1 + E) \ln(1 + E) \text{ or } \ln 1 + E \]

Our goal is to find the explicit solution for \(Y(x) \) and then evaluate it at \(x = 1 \), i.e., compute \(Y(1) \). This problem involves understanding the structure of the differential equation, applying appropriate methods for solving it, and carefully handling the initial conditions.

Understanding the Differential Equation and Initial Conditions

1. Analyzing the Equation \(Y y' = Xe \)

The given differential equation is:

\[Y y' = Xe \]

Here, \(Y \) is a function of \(x \), and \(y' \) denotes the derivative of \(y \) with respect to \(x \). The notation suggests that \(Y \) and \(y \) may be related, but clarity is necessary.

However, it's possible that there is a typographical confusion in the problem statement, and the intended differential equation is:

\[y y' = x e \]

or perhaps,

\[y y' = x e^x \]

Given the context and common differential equations, the most plausible form is:

$$ y y' = x e^x $$

which is a standard first-order differential equation suitable for solving via substitution.

Alternatively, if the original notation is different, we'll interpret the problem as:

$$ y y' = x e^x $$

with initial condition:

$$ y(0) = \frac{1}{2} \ln(1 + E) $$

where E might be Euler's number e.

Given the complexity and typical form, we proceed with the assumption that the differential equation is:

$$ y y' = x e^x $$

and the initial condition:

$$ y(0) = \frac{1}{2} \ln(1 + e) $$

Step-by-Step Solution of the Differential Equation

2. Recognizing the Type of Equation

The differential equation:

$$ y y' = x e^x $$

can be rewritten as:

$$ y \frac{dy}{dx} = x e^x $$

This is a separable differential equation because it can be expressed as:

$$ y dy = x e^x dx $$

which allows us to integrate both sides independently.

3. Separating Variables and Integrating

To solve, we explicitly separate the variables:

$$ y dy = x e^x dx $$

Now, integrate both sides:

$$ \int y dy = \int x e^x dx $$

The integral on the left is straightforward:

$$ \int y dy = \frac{1}{2} y^2 + C_1 $$

The integral on the right, $ \int x e^x dx $, requires integration by parts.

Computing the Integral $ \int x e^x dx $

4. Integration by Parts

Recall the integration by parts formula:

$$ \int u dv = uv - \int v du $$

Choose:

- $ u = x $ ⇒ $ du = dx $
- $ dv = e^x dx $ ⇒ $ v = e^x $

Applying the formula:

$$ \int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + C $$

Thus,

$$ \int x e^x dx = e^x (x - 1) + C $$

Formulating the General Solution

Putting everything together, the integrated form is:

$$ \frac{1}{2} y^2 = e^x (x - 1) + C $$

or equivalently,

$$
y^2 = 2 e^x (x - 1) + 2 C
$$

Let’s denote the constant $2 C$ as K:

$$
y^2 = 2 e^x (x - 1) + K
$$

To solve for y:

$$
y(x) = \pm \sqrt{2 e^x (x - 1) + K}
$$

Applying Initial Conditions to Find K

5. Initial Condition at $x = 0$

Given:

$$ y(0) = \frac{1}{2} \ln(1 + e) $$

Calculate $y(0)$:

$$
y(0) = \frac{1}{2} \ln(1 + e)
$$

Now, substitute $x = 0$ into the general solution to find K:

$$
\left(\frac{1}{2} \ln(1 + e) \right)^2 = 2 e^0 (0 - 1) + K
$$

Calculate each term:

- $e^0 = 1$
- $2 \times 1 \times (-1) = -2$

So:

$$
\left(\frac{1}{2} \ln(1 + e) \right)^2 = -2 + K
$$

Calculate the left side:

$$
\left(\frac{1}{2} \ln(1 + e) \right)^2 = \frac{1}{4} \left(\ln(1 + e) \right)^2
$$

Therefore:

$$
\frac{1}{4} \left(\ln(1 + e) \right)^2 = -2 + K
$$

Solve for K:

$$
K = \frac{1}{4} \left(\ln(1 + e) \right)^2 + 2
$$

Explicit Solution for $y(x)$

Putting everything together, the solution is:

$$
\boxed{
y(x) = \pm \sqrt{2 e^x (x - 1) + \frac{1}{4} \left(\ln(1 + e) \right)^2 + 2}
}
$$

Since the initial value $y(0)$ is positive, and the square root must be positive to satisfy the initial condition, we take the positive root:

$$
\boxed{
Y(x) = \sqrt{2 e^x (x - 1) + \frac{1}{4} \left(\ln(1 + e) \right)^2 + 2}
}
$$

\]

Evaluating \(Y(1) \)

Now, our goal is to find \(Y(1) \), i.e., the value of \(y \) at \(x = 1 \).

Substitute \(x = 1 \):

\[
Y(1) = \sqrt{2 e^1 (1 - 1) + \frac{1}{4} \left(\ln(1 + e) \right)^2 + 2}
\]

Calculate each term:

- \(e^1 = e \)
- \(1 - 1 = 0 \)
- \(2 e \times 0 = 0 \)

So, the expression simplifies to:

\[
Y(1) = \sqrt{0 + \frac{1}{4} \left(\ln(1 + e) \right)^2 + 2}
\]

which simplifies further to:

\[
Y(1) = \sqrt{\frac{1}{4} \left(\ln(1 + e) \right)^2 + 2}
\]

Alternatively, write as:

\[
Y(1) = \sqrt{\frac{1}{4} \left(\ln(1 + e) \right)^2 + 2}
\]

This is the exact value of \(Y(1) \).

Conclusion and Summary

6. Key Takeaways

- The initial differential equation $y y' = x e^x$ was separable, enabling straightforward integration.
- Integration by parts played a crucial role in evaluating $\int x e^x dx$.
- The initial condition allowed us to determine the constant K explicitly.
- The explicit solution for $y(x)$ was obtained, involving a square root function.
- Evaluating at $x=1$ yielded a precise expression for $Y(1)$.

7. Final Expression for $Y(1)$:

$$
\boxed{
Y(1) = \sqrt{\frac{1}{4} \left(\ln(1 + e) \right)^2 + 2}
}
$$

which can be approximated numerically if needed.

Additional Notes and Recommendations

- When solving differential equations involving logs

Frequently Asked Questions

What is the initial value problem given in the differential equation Yy = Xe, with Y(0) = 1?

The problem is to solve the differential equation Y y = X e with the initial condition Y(0) = 1 and find the value of Y at X=1.

How do we interpret the differential equation Y y = Xe?

It appears to be a typo or shorthand; likely it means Y' = X e, where Y' is the derivative of Y with respect to X, making it a first-order differential equation to solve.

What steps are involved in solving the differential equation Y' = X e?

Integrate both sides with respect to X: $Y = \int X e \, dx + C$, then determine the constant C using the initial condition $Y(0) = 1$.

How do we compute the integral of X e dx?

Use integration by parts: Let $u = X$, $dv = e \, dx$; then $du = dx$, $v = e$. So, $\int X e \, dx = X e - \int e \, dx = X e - e + C$.

What is the explicit solution for Y(X) after integrating and applying initial conditions?

$Y(X) = X e - e + C$. Applying $Y(0) = 1$ gives: $1 = 0 - e + C \Rightarrow C = 1 + e$. So, $Y(X) = X e - e + 1 + e = X e + 1$.

What is the value of Y(1) based on the solution Y(X) = X e + 1?

$Y(1) = 1 e + 1 = e + 1$.

Are there any other interpretations of the original problem involving logarithms or natural logs?

The original notation includes expressions like 'In(1 + E)' and 'Ln 1 + E', which may suggest involvement of natural logarithms. However, based on the differential equation provided, the straightforward solution yields $Y(1) = e + 1$. If logarithmic terms are involved, further clarification of the problem statement is needed.

Additional Resources

Differential Equations

Understanding and solving differential equations is fundamental in mathematical modeling, physics, engineering, and many applied sciences. They provide a way to describe how quantities change and interact over time or space. In this article, we delve deeply into a specific initial value problem (IVP):

$$ y y' = x e, \quad y(0) = 1, \quad \text{and find } y(1). $$

We'll explore the problem step-by-step, applying methods of solving differential equations, and ultimately compute the value of y at $x=1$.

Setting the Stage: The Initial Value Problem

What is an Initial Value Problem?

An initial value problem involves a differential equation coupled with specified values of the unknown function at a certain point. In our case, the differential equation is:
$$ y y' = x e, $$
with the initial condition:
$$ y(0) = 1. $$

Our goal is twofold:
1. Solve the differential equation for $y(x)$.
2. Determine the specific value $y(1)$.

Understanding the Differential Equation

The Equation $y y' = x e$

This is a first-order nonlinear differential equation because of the product $y y'$. To approach it, we need to manipulate it into a more manageable form.

Recognizing the Structure

Notice that $y y'$ can be viewed as $\frac{1}{2} (y^2)'$, since:
$$ \frac{d}{dx} (y^2) = 2 y y'. $$

This insight is crucial because it allows us to rewrite the differential equation in terms of the derivative of y^2.

Transforming the Equation: From Nonlinear to Separable

Step 1: Rewrite Using Chain Rule

Given:
$$ y y' = x e. $$
Multiplying both sides by 2:
$$ 2 y y' = 2 x e. $$
Recognize that:

$$\frac{d}{dx} (y^2) = 2 y y'.$$
So, the equation becomes:
$$\frac{d}{dx} (y^2) = 2 x e.$$

Step 2: Integrate Both Sides

Integrate both sides with respect to x:
$$y^2 = \int 2 x e \, dx + C,$$
where C is the constant of integration.

Step 3: Computing the Integral

Calculate $\int 2 x e \, dx$:

- Since e is a constant, we can factor it out:
$$2 e \int x \, dx.$$
- Integrate x:
$$\int x \, dx = \frac{1}{2} x^2.$$
- Therefore:
$$\int 2 x e \, dx = 2 e \times \frac{1}{2} x^2 = e x^2.$$

Step 4: Write the General Solution

Putting it all together:
$$y^2 = e x^2 + C.$$

Applying the Initial Condition

Step 1: Use the Condition $y(0) = 1$

At $x=0$, $y(0) = 1$, so:
$$(1)^2 = e \times 0^2 + C \Rightarrow 1 = 0 + C \Rightarrow C = 1.$$

Step 2: Final Expression for $y(x)$

Thus, the specific solution is:
$$y^2 = e x^2 + 1.$$
Taking the positive root (since $y(0) = 1 > 0$, and the initial condition suggests positive y near zero):
$$y(x) = \sqrt{e x^2 + 1}.$$

Finding $y(1)$: Evaluating at $x=1$

Step 1: Plug $x=1$ into the solution

$$ y(1) = \sqrt{e \times 1^2 + 1} = \sqrt{e + 1}. $$

Step 2: Numerical Approximation (Optional)

Given $e \approx 2.71828$,
$$ y(1) \approx \sqrt{2.71828 + 1} = \sqrt{3.71828} \approx 1.928. $$

Final answer:
$$
\boxed{
y(1) = \sqrt{e + 1}
}
$$

Summary and Insights

Why This Solution Works
The key step was recognizing $y y'$ as $\frac{1}{2} (y^2)'$, which simplified the nonlinear differential equation into a straightforward integral. This approach exemplifies how identifying derivatives of composite functions can streamline solving complex equations.

Broader Context
This problem demonstrates the power of substitution to transform nonlinear equations into linear or separable forms, making them accessible to standard calculus techniques. It also highlights the importance of initial conditions in determining the integration constant, ensuring the solution aligns with the problem's constraints.

Practical Implications
In real-world applications, such as physics or engineering, the ability to solve initial value problems precisely allows for accurate predictions of system behavior, stability analysis, and design optimization. This specific solution, involving exponential and polynomial expressions, appears in contexts like population dynamics, heat transfer, or electrical circuits where such differential relations naturally occur.

Conclusion

Solving the initial value problem $y y' = x e$, with $y(0) = 1$, and finding $y(1)$, showcases a classic technique in differential equations: transforming a nonlinear equation into an integrable form via recognizing derivatives of composite functions. The solution,

$$y(x) = \sqrt{e x^2 + 1},$$

not only satisfies the differential equation and initial condition but also emphasizes the elegance and power of calculus in solving real-world mathematical problems.

Understanding these methods equips students and professionals alike with a robust toolkit for tackling a wide range of differential equations encountered across scientific disciplines.

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