Use Variation Of Parameters To Solve The Given Nonhomogeneous System.dx/dt = 2x-ydy/dt = 3x-2y+6t

Use Variation Of Parameters To Solve The Given Nonhomogeneous System.dx/dt = 2x - y, dy/dt = 3x - 2y + 6t

When tackling systems of differential equations, especially nonhomogeneous ones, the method of variation of parameters often provides an effective approach. This technique extends the method of solving homogeneous systems by allowing parameters to vary with the independent variable, enabling us to find particular solutions to nonhomogeneous systems. In this article, we will explore step-by-step how to apply variation of parameters to solve the given nonhomogeneous system:

$$
\frac{dx}{dt} = 2x - y
$$

$$
\frac{dy}{dt} = 3x - 2y + 6t
$$

Understanding the process requires familiarity with systems of linear differential equations, methods for solving homogeneous systems, and the principles behind variation of parameters.

Understanding the System of Differential Equations

Before applying the variation of parameters, it’s essential to comprehend the structure of the system:

- It is a first-order linear system with constant coefficients and a nonhomogeneous term involving t.
- The system can be written in matrix form as:

$$
\frac{d}{dt}
\begin{bmatrix}
x \\
y
\end{bmatrix}
=
A
\begin{bmatrix}
x \\
y
\end{bmatrix}
+ \mathbf{f}(t)
$$

\]

where

\[
A =
\begin{bmatrix}
2 & -1 \\
3 & -2
\end{bmatrix}
,\quad
\mathbf{f}(t) =
\begin{bmatrix}
0 \\
6t
\end{bmatrix}
\]

Our goal is to find the complete solution, which is the sum of the general solution to the homogeneous system and a particular solution to the nonhomogeneous system.

Step 1: Solve the Homogeneous System

The homogeneous system is:

\[
\frac{d}{dt}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= A
\begin{bmatrix}
x \\
y
\end{bmatrix}
\]

or explicitly:

\[
\frac{dx}{dt} = 2x - y
\]
\[
\frac{dy}{dt} = 3x - 2y
\]

Find the Eigenvalues and Eigenvectors of Matrix A

- The characteristic equation:

\[
\det(A - \lambda I) = 0
\]

which is:

\[
\det
\begin{bmatrix}
2 - \lambda & -1 \\
3 & -2 - \lambda
\end{bmatrix}
= (2 - \lambda)(-2 - \lambda) - (-1)(3)
\]

Calculating:

\[
(2 - \lambda)(-2 - \lambda) + 3
\]

\[
= -4 - 2\lambda + 2\lambda + \lambda^2 + 3
\]

\[
= \lambda^2 - 1
\]

- Set equal to zero:

\[
\lambda^2 - 1 = 0 \Rightarrow \lambda^2 = 1 \Rightarrow \lambda = \pm 1
\]

Eigenvectors Corresponding to Eigenvalues

- For \(\lambda = 1\):

\[
(A - I) \mathbf{v} = 0
\]

\[

\begin{bmatrix}
2 - 1 & -1 \\
3 & -2 - 1
\end{bmatrix}
=
\begin{bmatrix}
1 & -1 \\
3 & -3
\end{bmatrix}
\]

- From the first row:

\[
1 \cdot v_1 - 1 \cdot v_2 = 0 \Rightarrow v_1 = v_2
\]

- Choose \(v_2 = 1\), then \(v_1 = 1\)

- Eigenvector:

\[
\mathbf{v}_1 =
\begin{bmatrix}
1 \\
1
\end{bmatrix}
\]

- For \(\lambda = -1\):

\[
A + I =
\begin{bmatrix}
2 + 1 & -1 \\
3 & -2 + 1
\end{bmatrix}
=
\begin{bmatrix}
3 & -1 \\
3 & -1
\end{bmatrix}
\]

- From the first row:

\[
3 v_1 - v_2 = 0 \Rightarrow v_2 = 3 v_1
\]

- Choose \(v_1 = 1\), then \(v_2 = 3\)

- Eigenvector:

$$
\mathbf{v}_2 =
\begin{bmatrix}
1 \\
3
\end{bmatrix}
$$

Homogeneous Solution

The general homogeneous solution:

$$
\mathbf{x}_h(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2
$$

becomes:

$$
\boxed{
\begin{bmatrix}
x_h(t) \\
y_h(t)
\end{bmatrix}
= C_1 e^{t}
\begin{bmatrix}
1 \\
1
\end{bmatrix}
+ C_2 e^{-t}
\begin{bmatrix}
1 \\
3
\end{bmatrix}
}
$$

where C_1, C_2 are arbitrary constants.

Step 2: Find a Particular Solution Using Variation of Parameters

The method of variation of parameters involves replacing the constants C_1, C_2 with functions $u_1(t)$, $u_2(t)$:

$$
\mathbf{x}_p(t) = u_1(t) e^{t} \mathbf{v}_1 + u_2(t) e^{-t} \mathbf{v}_2
$$

which leads us to set:

$$
\begin{bmatrix}
x_p(t) \\
y_p(t)
\end{bmatrix}
= u_1(t) e^{t}
\begin{bmatrix}
1 \\
1
\end{bmatrix}
+ u_2(t) e^{-t}
\begin{bmatrix}
1 \\
3
\end{bmatrix}
$$

To find $u_1(t)$ and $u_2(t)$, we impose the condition:

$$
u_1'(t) e^{t} \mathbf{v}_1 + u_2'(t) e^{-t} \mathbf{v}_2 = \mathbf{0}
$$

which simplifies the derivatives, leading to the system:

$$
\begin{bmatrix}
e^{t} & e^{-t} \\
e^{t} & 3 e^{-t}
\end{bmatrix}
\begin{bmatrix}
u_1'(t) \\
u_2'(t)
\end{bmatrix}
=
\begin{bmatrix}
0 \\
6t
\end{bmatrix}
$$

Step 3: Solve for $u_1'(t)$ and $u_2'(t)$

The system:

$$
\begin{bmatrix}
e^{t} & e^{-t} \\
e^{t} & 3 e^{-t}
\end{bmatrix}
\begin{bmatrix}
u_1' \\
u_2'
\end{bmatrix}
=
\begin{bmatrix}
0 \\
6t
\end{bmatrix}
$$

has a determinant:

$$
D = e^{t} \cdot 3 e^{-t} - e^{t} \cdot e^{-t} = 3 - 1 = 2
$$

Using Cramer's rule:

$$
u_1' = \frac{1}{D}
\begin{bmatrix}
0 & e^{-t} \\
6t & 3 e^{-t}
\end{bmatrix}
$$

but more straightforwardly, the formulas:

$$
u_1' = \frac{1}{D} \left((0) \cdot 3 e^{-t} - (6t) \cdot e^{-t} \right) = - \frac{6 t e^{-t}}{2} = -3 t e^{-t}
$$

$$
u_2' = \frac{1}{D} \left(e^{t} \cdot 6 t - 0 \cdot e^{t} \right) = \frac{6 t e^{t}}{2} = 3 t e^{t}
$$

Step 4: Integrate to Find $u_1(t)$ and $u_2(t)$

- For $u_1(t)$:

$$
u_1(t) = \int u_1'(t) dt = \int -3 t e^{-t} dt
$$

- For \(u

Frequently Asked Questions

What is the general approach to solving the nonhomogeneous system using variation of parameters?

The method involves first solving the associated homogeneous system to find the fundamental matrix, then assuming particular solutions with variable parameters, and finally determining these parameters by substituting back into the system to satisfy the nonhomogeneous equations.

How do we find the fundamental matrix for the homogeneous part of the system?

Solve the homogeneous system $dx/dt = 2x - y$, $dy/dt = 3x - 2y$, by expressing it in matrix form and finding the eigenvalues and eigenvectors to construct the fundamental matrix.

What are the eigenvalues and eigenvectors of the system's coefficient matrix?

The coefficient matrix is [[2, -1], [3, -2]]. Its eigenvalues are $\lambda=0$ and $\lambda=-2$, with corresponding eigenvectors $v1 = [1, 1]$ for $\lambda=0$, and $v2 = [1, -1]$ for $\lambda=-2$.

How is the particular solution constructed using variation of parameters?

Assume particular solutions of the form $x_p(t) = u1(t)v1 + u2(t)v2$ and $y_p(t)$ similarly, where $u1(t)$ and $u2(t)$ are functions to be determined by plugging into the system and solving the resulting equations.

What are the steps to determine the functions u1(t) and u2(t) in the variation of parameters method?

Differentiate the assumed particular solution, set up the system for $du1/dt$ and $du2/dt$ based on the nonhomogeneous terms, and solve these equations, often using the Wronskian or matrix methods.

How do the nonhomogeneous terms (like 6t in dy/dt) affect the variation of parameters solution?

They act as forcing functions that influence the equations for u1(t) and u2(t), leading to integrals involving these nonhomogeneous terms when solving for the functions u1(t) and u2(t), ultimately affecting the particular solution.

What is the role of the Wronskian in applying variation of parameters to this system?

The Wronskian, derived from the fundamental matrix, helps in solving for the derivatives of the unknown functions u1(t) and u2(t), ensuring the system for these functions is invertible and solvable.

After finding the particular solution, how do you write the general solution to the system?

The general solution is the sum of the homogeneous solution (from the fundamental matrix) and the particular solution obtained via variation of parameters: ($X(t) = X_h(t) + X_p(t)$).

Are there any special considerations when applying variation of parameters to systems with time-dependent nonhomogeneous terms?

Yes, the integrals involved may be more complex, and sometimes they require special techniques or substitutions; ensuring the integrability and handling any potential singularities are also important.

Additional Resources

Use Variation Of Parameters To Solve The Given Nonhomogeneous System

In the realm of differential equations, systems of linear equations hold significant importance across various scientific and engineering disciplines. Among the myriad techniques employed to solve these systems, the method of variation of parameters stands out as a powerful and elegant approach, especially when tackling nonhomogeneous systems. This article delves into the application of variation of parameters to solve a specific nonhomogeneous system:

$$
\frac{dx}{dt} = 2x - y \quad \text{and} \quad \frac{dy}{dt} = 3x - 2y + 6t.
$$

Through an in-depth exploration, we will examine the theoretical underpinnings, step-by-step solution methodology, and interpretative insights that underscore the utility of variation of parameters in such contexts.

Understanding the System and Its Structure

The system under consideration is:

$$
\begin{cases}
\frac{dx}{dt} = 2x - y, \\
\frac{dy}{dt} = 3x - 2y + 6t.
\end{cases}
$$

Rewriting in matrix form:

$$
\mathbf{X}'(t) = A \mathbf{X}(t) + \mathbf{f}(t),
$$

where

$$
\mathbf{X}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}, \quad
A = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}, \quad
\mathbf{f}(t) = \begin{bmatrix} 0 \\ 6t \end{bmatrix}.
$$

This form clarifies that the system is a nonhomogeneous linear system. To solve it, one typically decomposes the problem into solving the associated homogeneous system and then finding a particular solution to the nonhomogeneous part.

Decomposition: Homogeneous System and Particular Solution

The Homogeneous System

The homogeneous counterpart is:

$$
\mathbf{X}_h'(t) = A \mathbf{X}_h(t),
$$

which simplifies to:

\[

\begin{cases}
\frac{dx}{dt} = 2x - y, \\
\frac{dy}{dt} = 3x - 2y.
\end{cases}
\]

The solution hinges on finding the eigenvalues and eigenvectors of matrix \(A\).

Eigenvalues of \(A\)

The characteristic equation:

\[
\det(A - \lambda I) = 0,
\]

becomes:

\[
\det \begin{bmatrix} 2 - \lambda & -1 \\ 3 & -2 - \lambda \end{bmatrix} = 0,
\]

which simplifies to:

\[
(2 - \lambda)(-2 - \lambda) - (-1)(3) = 0,
\]

\[
(2 - \lambda)(-2 - \lambda) + 3 = 0.
\]

Expanding:

\[
-(2 - \lambda)(2 + \lambda) + 3 = 0,
\]

\[
-[(2)(2) + 2\lambda - 2\lambda - \lambda^2] + 3= 0,
\]

\[
- (4 - \lambda^2) + 3= 0,
\]

\[
-4 + \lambda^2 + 3 = 0,
\]

$$
\lambda^2 - 1 = 0,
$$

$$
\lambda^2 = 1,
$$

$$
\lambda = \pm 1.
$$

Thus, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -1$.

Eigenvectors

- For $\lambda = 1$:

Solve $(A - I)\mathbf{v} = 0$:

$$
\begin{bmatrix} 2 - 1 & -1 \\ 3 & -2 - 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}.
$$

From the first row:

$$
1 \cdot v_1 - 1 \cdot v_2 = 0 \Rightarrow v_1 = v_2.
$$

Choose $v_1 = 1$, then $v_2=1$. Eigenvector:

$$
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
$$

- For $\lambda = -1$:

Solve $(A + I)\mathbf{v} = 0$:

$$
\begin{bmatrix} 2 + 1 & -1 \\ 3 & -2 + 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}.
$$

From the first row:

$$

$$
3 v_1 - v_2= 0 \Rightarrow v_2= 3 v_1.
$$

Choose $v_1=1$, then $v_2=3$. Eigenvector:

$$
\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.
$$

Homogeneous Solution

The general homogeneous solution:

$$
\mathbf{X}_h(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2,
$$

which explicitly is:

$$
\begin{bmatrix} x_h(t) \\ y_h(t) \end{bmatrix} = c_1 e^{t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ 3 \end{bmatrix}.
$$

Particular Solution via Variation of Parameters

The core challenge lies in finding a particular solution $\mathbf{X}_p(t)$ that satisfies the nonhomogeneous system:

$$
\mathbf{X}'(t) = A \mathbf{X}(t) + \mathbf{f}(t).
$$

Methodology Overview

The variation of parameters approach involves:

- Using the fundamental matrix $\Phi(t)$ constructed from the homogeneous solutions.
- Allowing the constants c_1, c_2 to vary as functions $u_1(t), u_2(t)$.
- Deriving a system of equations for $u_1(t)$ and $u_2(t)$ that ensures the particular solution satisfies the original system.

Constructing the Fundamental Matrix $\Phi(t)$

$$
\Phi(t) = \left[\mathbf{v}_1 e^{\lambda_1 t} \quad \mathbf{v}_2 e^{\lambda_2 t}\right] = \begin{bmatrix} e^t & e^{-t} \\ e^t & 3 e^{-t} \end{bmatrix}.
$$

Note: The columns are the fundamental solutions.

Set Up for Variation of Parameters

Let:

$$
\mathbf{X}_p(t) = \Phi(t) \mathbf{u}(t),
$$

where

$$
\mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.
$$

Differentiating:

$$
\mathbf{X}_p' = \Phi'(t) \mathbf{u} + \Phi(t) \mathbf{u}',
$$

and substituting into the original system:

$$
\Phi'(t) \mathbf{u} + \Phi(t) \mathbf{u}' = A \Phi(t) \mathbf{u} + \mathbf{f}(t).
$$

Since $\Phi'(t) = A \Phi(t)$, this simplifies to:

$$
A \Phi(t) \mathbf{u} + \Phi(t) \mathbf{u}' = A \Phi(t) \mathbf{u} + \mathbf{f}(t),
$$

which implies:

$$
\Phi(t) \mathbf{u}' = \mathbf{f}(t).
$$

This is a key step; it reduces to solving:

\[
\Phi(t) \mathbf{u}' = \mathbf{f}(t).
\]

Solving for \(\mathbf{u}(t)\)

Explicitly,

\[
\begin{bmatrix}
e^t & e^{-t} \\
e^t & 3 e^{-t}
\end{bmatrix}
\begin{bmatrix}
u_1'(t) \\
u_2'(t)
\end{bmatrix}
=
\begin{bmatrix}
0 \\
6t
\end{bmatrix}.
\]

This leads to the system:

\[
\begin{cases}
e^t u_1' + e^{-t} u_2' = 0,

Use Variation Of Parameters To Solve The Given Nonhomogeneous System Dx Dt 2x Ydy Dt 3x 2y 6t

Use Variation Of Parameters To Solve The Given Nonhomogeneous System Dx Dt 2x Ydy Dt 3x 2y 6t

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