1. Find The First Fundamental Form For The Spherical Coordinate Of The Unit Spherical Surface2. Find

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Understanding the geometry of surfaces is fundamental in differential geometry, especially when analyzing properties like curvature and geodesics. One key concept in this study is the first fundamental form, which encapsulates the metric properties of a surface, such as lengths, angles, and areas. This article provides a comprehensive guide to deriving the first fundamental form for the spherical coordinate representation of the unit sphere. We will explore the parametrization of the sphere, derive the metric components, and discuss the significance of the first fundamental form in geometric analysis.

Introduction to the Unit Sphere and Spherical Coordinates

Before delving into the derivation of the first fundamental form, it is essential to understand the setting:

- The unit sphere, denoted as S^2, is the set of all points in $\mathbb{R}^3$ at a distance of 1 from the origin:

$$
S^2 = \left\{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \right\}
$$

- Spherical coordinates provide a natural parametrization of the sphere, typically denoted as (θ, ϕ), where:

- θ (colatitude) measures the angle from the positive z-axis, $0 \leq \theta \leq \pi$.
- ϕ (longitude) measures the angle from the positive x-axis in the xy-plane, $0 \leq \phi < 2\pi$.

The standard parametrization of the unit sphere in spherical coordinates is:

$$
\boxed{
\mathbf{r}(\theta, \phi) = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)
}
$$

This parametrization maps the domain $(\theta, \phi) \in [0, \pi] \times [0, 2\pi)$ onto the sphere.

Deriving the First Fundamental Form

The first fundamental form I encapsulates the metric properties on a surface and is derived from the inner products of the tangent vectors to the surface. It is expressed in terms of the parameters as:

$$
I = E\, d\theta^2 + 2F\, d\theta\, d\phi + G\, d\phi^2
$$

where:

- $E = \left\langle \mathbf{r}_\theta, \mathbf{r}_\theta \right\rangle$,
- $F = \left\langle \mathbf{r}_\theta, \mathbf{r}_\phi \right\rangle$,
- $G = \left\langle \mathbf{r}_\phi, \mathbf{r}_\phi \right\rangle$.

Here, $\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta}$ and $\mathbf{r}_\phi = \frac{\partial \mathbf{r}}{\partial \phi}$.

Step 1: Compute the Partial Derivatives

Calculate $\mathbf{r}_\theta$ and $\mathbf{r}_\phi$:

$$
\mathbf{r}_\theta = \frac{\partial}{\partial \theta} (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = (\cos \theta \cos \phi, \cos \theta \sin \phi, -\sin \theta)
$$

$$
\mathbf{r}_\phi = \frac{\partial}{\partial \phi} (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = (-\sin \theta \sin \phi, \sin \theta \cos \phi, 0)
$$

Step 2: Calculate the Metric Components

Using the inner product in $\mathbb{R}^3$, note that:

$$
\left\langle \mathbf{a}, \mathbf{b} \right\rangle = a_x b_x + a_y b_y + a_z b_z
$$

Compute each component:

- E:

$$
E = \left\langle \mathbf{r}_\theta, \mathbf{r}_\theta \right\rangle = (\cos \theta \cos \phi)^2 + (\cos \theta \sin \phi)^2 + (-\sin \theta)^2
$$

$$
E = \cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \sin^2 \theta
$$

Using $\cos^2 \phi + \sin^2 \phi = 1$:

$$
E = \cos^2 \theta (1) + \sin^2 \theta = \cos^2 \theta + \sin^2 \theta = 1
$$

- F:

$$
F = \left\langle \mathbf{r}_\theta, \mathbf{r}_\phi \right\rangle = (\cos \theta \cos \phi)(-\sin \theta \sin \phi) + (\cos \theta \sin \phi)(\sin \theta \cos \phi) + (-\sin \theta)(0)
$$

Simplify:

$$
F = -\cos \theta \sin \theta \cos \phi \sin \phi + \cos \theta \sin \theta \sin \phi \cos \phi + 0
$$

Note that these two terms cancel each other:

$$
F = 0
$$

- G:

$$
G = \left\langle \mathbf{r}_\phi, \mathbf{r}_\phi \right\rangle = (-\sin \theta \sin \phi)^2 + (\sin \theta \cos \phi)^2 + 0^2
$$

$$
G = \sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi = \sin^2 \theta (\sin^2 \phi + \cos^2 \phi) = \sin^2 \theta
$$

Final Expression for the First Fundamental Form

Putting it all together, the metric components are:

$$
E = 1, \quad F = 0, \quad G = \sin^2 \theta
$$

Therefore, the first fundamental form (the metric tensor) on the unit sphere in spherical coordinates is:

$$
\boxed{
I = d\theta^2 + \sin^2 \theta\, d\phi^2
}
$$

This expression shows that the metric on the unit sphere is conformally equivalent to the standard metric, scaled by $\sin^2 \theta$ in the ϕ-direction.

Significance of the First Fundamental Form in Differential Geometry

The first fundamental form is central to understanding the intrinsic geometry of surfaces. Its importance includes:

- Measuring lengths: For a tangent vector $\mathbf{v} = a\, \partial_\theta + b\, \partial_\phi$, the length squared is:

$$
|\mathbf{v}|^2 = E a^2 + 2F a b + G b^2 = a^2 + \sin^2 \theta\, b^2
$$

- Calculating angles: The angle between two tangent vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is derived from the inner product defined by the first fundamental form.

- Determining areas: The area element dA on the sphere is:

$$
dA = \sqrt{\det(g_{ij})}\, d\theta\, d\phi = \sin \theta\, d\theta\, d\phi
$$

where g_{ij} is the metric tensor.

- Studying geodesics: The shortest paths on the sphere are determined by the metric, leading to great circles.

Applications and Further Implications

Understanding the first fundamental form on the sphere facilitates numerous applications:

- Geodesic computation: Calculating shortest paths on the sphere, essential in navigation and geodesy.
- Surface area calculations: Deriving the surface area of spherical segments.
- Curvature analysis: The first fundamental form feeds into the second fundamental form to compute Gaussian and mean curvature.
- Physics and engineering: In fields like general relativity, the metric defines spacetime geometry. On the sphere, it models planetary surfaces, celestial mechanics, and more.

Conclusion

The derivation of the first fundamental form for the spherical coordinate parametrization of the unit sphere reveals a simple yet profound metric:

$$
I = d\theta^2 + \sin^2 \theta\, d\phi^2
$$

This form encapsulates the intrinsic geometry of the sphere, enabling precise measurements of lengths, angles, and areas. Recognizing the structure of the metric allows

Frequently Asked Questions

What is the first fundamental form in spherical coordinates on a unit sphere?

The first fundamental form on a unit sphere in spherical coordinates (θ, ϕ) is given by $ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$, representing the metric for the surface.

How do you derive the first fundamental form for the spherical coordinate system on a sphere?

By parameterizing the sphere as $r(\theta, \phi) = (\sin\theta \cos\phi, \sin\theta \sin\phi, \cos\theta)$, then computing the dot products of partial derivatives with respect to θ and ϕ, leading to the metric components.

What are the parameter ranges for spherical coordinates on the unit sphere?

Typically, θ ranges from 0 to π (polar angle), and ϕ ranges from 0 to 2π (azimuthal angle).

Why is the first fundamental form important in differential geometry?

It encodes the intrinsic geometry of a surface, allowing calculation of lengths, angles, and areas directly from the metric without embedding in higher-dimensional space.

Can you explain the significance of the coefficients in the first fundamental form for a sphere?

Yes, the coefficients (E, F, G) in $ds^2 = E\, d\theta^2 + 2F\, d\theta\, d\phi + G\, d\phi^2$ describe how lengths and angles are measured on the surface; for a sphere, typically $E=1$, $F=0$, $G=\sin^2\theta$.

How does the first fundamental form change if we consider a different radius for the sphere?

If the sphere has radius R, the first fundamental form becomes $ds^2 = R^2\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2$, scaling the metric components accordingly.

What is the role of the metric tensor in finding the first fundamental form?

The metric tensor components (g_{ij}) directly determine the coefficients of the first fundamental form, representing the inner products of tangent vectors.

How do you verify that the first fundamental form correctly represents the surface geometry?

By computing lengths of curves and angles between vectors using the form and comparing them with known geometric properties of the sphere.

What are some applications of understanding the first fundamental form on a spherical surface?

Applications include geodesy (Earth's surface modeling), computer graphics (texture mapping), and physics (studying curved spaces in general relativity).

Additional Resources

1. Find The First Fundamental Form For The Spherical Coordinate Of The Unit Spherical Surface2. Find

In the realm of differential geometry, understanding the intrinsic properties of surfaces is fundamental, especially when these surfaces are curved, such as spheres. The process of examining these properties often begins with the computation of the First Fundamental Form (FFF), a mathematical construct that encapsulates how distances and angles are measured on the surface itself, independent of how the surface is embedded in space. This article delves into the detailed derivation of the First Fundamental Form for the unit sphere expressed through spherical coordinates, paving the way for deeper insights into the geometry of spherical surfaces.

The Significance of the First Fundamental Form

Before embarking on the derivation, it's crucial to understand why the First Fundamental Form holds such importance in differential geometry. Essentially, it acts as the metric tensor for the surface, enabling us to compute:

- Distances between points on the surface,
- Angles between curves intersecting on the surface,
- Surface areas bounded by curves.

In the context of a sphere, understanding the FFF allows mathematicians and engineers to analyze phenomena such as geodesic paths, surface curvature, and even physical processes like wave propagation or gravitational effects on planetary surfaces.

Setting the Stage: Spherical Coordinates on the Unit Sphere

The unit sphere, denoted as S^2, is defined as the set of points in three-dimensional space satisfying:

$$
x^2 + y^2 + z^2 = 1
$$

To parametrize the surface, spherical coordinates are employed, typically represented as:

$$
\begin{cases}
x = \sin \theta \cos \phi \\
y = \sin \theta \sin \phi \\
z = \cos \theta
\end{cases}
$$

where:

- θ (polar angle) ranges from 0 to π,
- ϕ (azimuthal angle) ranges from 0 to 2π.

This parametrization maps every point on the sphere's surface to a pair (θ, ϕ), making it ideal for analyzing surface properties.

Deriving the First Fundamental Form

The first step in deriving the FFF is to compute the partial derivatives of the position vector $\mathbf{r}(\theta, \phi)$:

$$
\mathbf{r}(\theta, \phi) = \begin{bmatrix}
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\cos \theta
\end{bmatrix}
$$

Calculating Partial Derivatives

1. Partial derivative with respect to θ:

$$
\frac{\partial \mathbf{r}}{\partial \theta} =
\begin{bmatrix}
\cos \theta \cos \phi \\
\cos \theta \sin \phi \\
- \sin \theta
\end{bmatrix}
$$

2. Partial derivative with respect to ϕ:

$$
\frac{\partial \mathbf{r}}{\partial \phi} =
\begin{bmatrix}
- \sin \theta \sin \phi \\
\sin \theta \cos \phi \\
0
\end{bmatrix}
$$

Computing the Metric Coefficients

The First Fundamental Form is expressed as:

$$
ds^2 = E\, d\theta^2 + 2F\, d\theta d\phi + G\, d\phi^2
$$

where:

- $E = \left\langle \frac{\partial \mathbf{r}}{\partial \theta}, \frac{\partial \mathbf{r}}{\partial \theta} \right\rangle$,
- $F = \left\langle \frac{\partial \mathbf{r}}{\partial \theta}, \frac{\partial \mathbf{r}}{\partial \phi} \right\rangle$,
- $G = \left\langle \frac{\partial \mathbf{r}}{\partial \phi}, \frac{\partial \mathbf{r}}{\partial \phi} \right\rangle$.

Step-by-Step Calculation

1. Computing E:

$$
E = \left\langle \frac{\partial \mathbf{r}}{\partial \theta}, \frac{\partial \mathbf{r}}{\partial \theta} \right\rangle
= (\cos \theta \cos \phi)^2 + (\cos \theta \sin \phi)^2 + (- \sin \theta)^2
$$

Simplify:

$$
E = \cos^2 \theta \cos^2 \phi + \cos^2 \theta \sin^2 \phi + \sin^2 \theta
$$

Using $\cos^2 \phi + \sin^2 \phi = 1$:

$$
E = \cos^2 \theta (1) + \sin^2 \theta = \cos^2 \theta + \sin^2 \theta = 1
$$

2. Computing F:

$$
F = \left\langle \frac{\partial \mathbf{r}}{\partial \theta}, \frac{\partial \mathbf{r}}{\partial \phi} \right\rangle
= (\cos \theta \cos \phi)(- \sin \theta \sin \phi) + (\cos \theta \sin \phi)(\sin \theta \cos \phi) + (- \sin \theta)(0)
$$

Simplify:

$$
F = - \cos \theta \sin \theta \cos \phi \sin \phi + \cos \theta \sin \theta \sin \phi \cos \phi + 0
$$

Note that both terms are negatives of each other and cancel out:

$$
F = 0
$$

3. Computing G:

$$
G = \left\langle \frac{\partial \mathbf{r}}{\partial \phi}, \frac{\partial \mathbf{r}}{\partial \phi} \right\rangle
= (- \sin \theta \sin \phi)^2 + (\sin \theta \cos \phi)^2 + 0^2
$$

Simplify:

$$
G = \sin^2 \theta \sin^2 \phi + \sin^2 \theta \cos^2 \phi
$$

Again, using $\sin^2 \phi + \cos^2 \phi = 1$:

$$
G = \sin^2 \theta (1) = \sin^2 \theta
$$

\]

Final Expression of the First Fundamental Form

Putting everything together, the metric tensor components are:

$$
E = 1, \quad F = 0, \quad G = \sin^2 \theta
$$

Thus, the First Fundamental Form (or the surface element metric) for the unit sphere in spherical coordinates is:

$$
ds^2 = d\theta^2 + \sin^2 \theta\, d\phi^2
$$

This compact form succinctly captures the intrinsic geometry of the sphere: distances along the θ-direction are measured directly, while distances along the ϕ-direction are scaled by $\sin \theta$.

Implications and Applications

Understanding the FFF for the sphere is not merely a mathematical curiosity; it has profound applications across multiple disciplines:

- Geodesics: The shortest paths on the sphere, which are great circles, can be analyzed using the metric.
- Surface Area Calculations: Integrating the surface element $\sqrt{EG - F^2} \, d\theta d\phi$ yields the surface area.
- Physical Models: In geophysics, climate modeling, and astrophysics, the metric informs how phenomena propagate across planetary or stellar surfaces.
- Computer Graphics & Visualization: Accurate rendering of spherical surfaces depends on understanding their intrinsic geometry.

Broader Context: Curvature and Beyond

While the First Fundamental Form provides the metric properties, the broader study involves the Second Fundamental Form, which relates to the surface's curvature, and the Gauss–Codazzi equations, which tie the intrinsic and extrinsic geometry together. Mastery of these concepts allows mathematicians and scientists to analyze complex behaviors of curved surfaces, from the shape of planets to the fabric of spacetime.

Conclusion

Deriving the First Fundamental Form for the unit sphere in spherical coordinates reveals a simple yet powerful metric:

$$
ds^2 = d\theta^2 + \sin^2 \theta\, d\phi^2
$$

This formula is foundational in the study of spherical geometry, providing the basis for understanding distances, angles, and curvature on the surface. Its elegance underscores the beauty of differential geometry: even in the curved realm of the sphere, the intrinsic properties can be succinctly expressed, offering profound

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