(1 Point) Use Spherical Coordinates To Evaluate The Triple Integralee(x2 Y2 Z2)x2 Y2 Z2dv,where E Is

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Understanding the Problem: Evaluating a Triple Integral Using Spherical Coordinates

When tackling complex volume integrals in multivariable calculus, choosing the appropriate coordinate system simplifies the process immensely. The problem at hand involves evaluating the triple integral

$$
\iiint_E x^2 y^2 z^2 \, dv,
$$

where E is a specified region in three-dimensional space. The question hints at employing spherical coordinates to evaluate this integral, which is especially advantageous when E exhibits spherical symmetry or is bounded by spherical surfaces.

This article explores how to approach such an integral systematically, transforming from Cartesian to spherical coordinates, setting up the integral properly, and computing the integral step-by-step. Whether you're a student learning multivariable calculus or a professional dealing with volume integrations, understanding this process provides a powerful tool for solving a broad class of problems.

Introduction to Spherical Coordinates

What Are Spherical Coordinates?

Spherical coordinates (ρ, θ, ϕ) are a three-dimensional coordinate system where:

- ρ (radius) is the distance from the origin to the point.
- θ (azimuthal angle) is the angle in the xy-plane from the positive x-axis.
- ϕ (polar angle) is the angle from the positive z-axis downward

to the point.

The relationships between Cartesian coordinates (x, y, z) and spherical coordinates are:

$$
\begin{cases}
x = \rho \sin \phi \cos \theta, \\
y = \rho \sin \phi \sin \theta, \\
z = \rho \cos \phi,
\end{cases}
$$

with $\rho \geq 0$, $0 \leq \phi \leq \pi$, and $0 \leq \theta < 2\pi$.

The Jacobian for Spherical Coordinates

When changing variables in an integral, the volume element dv transforms as:

$$
dv = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.
$$

This Jacobian determinant accounts for the distortion of volume elements from Cartesian to spherical coordinates.

Step-by-Step Approach to Evaluating the Integral

1. Understanding the Region E

Before transforming the integral, you must understand the region E. For spherical coordinates, typical regions are bounded by spheres, cones, or other surfaces expressible conveniently in spherical coordinates.

Suppose, for example, E is the solid ball of radius R, i.e.,

$$
E = \{ (x, y, z) \mid x^2 + y^2 + z^2 \leq R^2 \}.
$$

In spherical coordinates, this corresponds to:

$$
0 \leq \rho \leq R, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi.
$$

You should adapt the limits depending on the actual description of E.

2. Expressing the Integrand in Spherical Coordinates

The integrand is:

$$
x^2 y^2 z^2.
$$

Substituting the spherical coordinate expressions:

$$
\begin{aligned}
x &= \rho \sin \phi \cos \theta, \\
y &= \rho \sin \phi \sin \theta, \\
z &= \rho \cos \phi.
\end{aligned}
$$

Thus,

$$
x^2 y^2 z^2 = (\rho \sin \phi \cos \theta)^2 \times (\rho \sin \phi \sin \theta)^2 \times (\rho \cos \phi)^2.
$$

Calculating step-by-step:

$$
x^2 = \rho^2 \sin^2 \phi \cos^2 \theta,
$$
$$
y^2 = \rho^2 \sin^2 \phi \sin^2 \theta,
$$
$$
z^2 = \rho^2 \cos^2 \phi.
$$

Multiplying all:

$$
x^2 y^2 z^2 = \rho^6 \sin^4 \phi \cos^2 \theta \sin^2 \theta \cos^2 \phi.
$$

3. Setting Up the Integral in Spherical Coordinates

The volume element:

$$
dv = \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta,
$$

so the integral becomes:

$$
\iiint_E x^2 y^2 z^2 \, dv = \int_{\theta} \int_{\phi} \int_{\rho} \left[\rho^6 \sin^4 \phi \cos^2 \theta \sin^2 \theta \cos^2 \phi \right] \times \left[\rho^2 \sin \phi \right] d\rho \, d\phi \, d\theta.
$$

Combining:

$$
= \int_{\theta} \int_{\phi} \int_{\rho} \rho^8 \sin^5 \phi \cos^2 \phi \times \cos^2 \theta \sin^2 \theta \, d\rho \, d\phi \, d\theta.
$$

The limits depend on the region E; for a sphere of radius R:

$$
0 \leq \rho \leq R, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta < 2\pi.
$$

4. Separating the Integral into Multiple Parts

Since the integrand factors into functions of ρ, ϕ, and θ separately, the triple integral can be written as a product of three integrals:

$$
\left(\int_0^R \rho^8 \, d\rho \right) \times \left(\int_0^\pi \sin^5 \phi \cos^2 \phi \, d\phi \right) \times \left(\int_0^{2\pi} \cos^2 \theta \sin^2 \theta \, d\theta \right).
$$

This separation simplifies the evaluation process significantly.

Evaluating Each Integral Component

1. Radial Integral

$$
I_\rho = \int_0^R \rho^8 \, d\rho = \frac{\rho^9}{9} \Big|_0^R = \frac{R^9}{9}.
$$

2. Angular Integral in ϕ

$$
I_\phi = \int_0^\pi \sin^5 \phi \cos^2 \phi \, d\phi.
$$

Use substitution to evaluate this integral:

- Let $u = \cos \phi$, then $du = -\sin \phi \, d\phi$,
- When $\phi = 0$, $u = 1$;
- When $\phi = \pi$, $u = -1$.

Express $\sin^5 \phi$ in terms of u:

$$
\sin^5 \phi = (1 - u^2)^{5/2}.
$$

Since $du = -\sin \phi \, d\phi$,

$$
\sin \phi \, d\phi = -du,
$$

then

$$
I_\phi = \int_{u=1}^{-1} (1 - u^2)^{5/2} u^2 \times \frac{\sin \phi}{\sin \phi} \, d\phi.
$$

Alternatively, it's simpler to write:

$$
I_\phi = \int_0^\pi \sin^5 \phi \cos^2 \phi \, d\phi,
$$

which can be tackled by expressing $\sin^5 \phi = \sin^4 \phi \sin \phi$, then using identities or reduction formulas.

Using the substitution $u = \cos \phi$:

$$
\sin \phi \, d\phi = -du,
$$

and

$$
\sin^4 \phi = (1 - u^2)^2,
$$

so:

$$
I_\phi = \int_{u=1}^{-1} (1 - u^2)^2 u^2 \times (-du) = \int_{-1}^1 (1 - u^2)^2 u^2 \, du.
$$

Expand:

$$
(1 - u^2)^2 = 1 - 2u^2 + u^4,
$$

then:

$$
I_\phi = \int_{-1}^1 u^2 (1
$$

Frequently Asked Questions

How do you set up the triple integral of (x^2 y^2 z^2) over a region E using spherical coordinates?

To set up the integral in spherical coordinates, express x, y, and z in terms of r, θ, and φ: $x = r \sin φ \cos θ$, $y = r \sin φ \sin θ$, $z = r \cos φ$. Then, substitute into the integrand: x^2 y^2 z^2 = $(r \sin φ \cos θ)^2$ $(r \sin φ \sin θ)^2$ $(r \cos φ)^2$, and include the Jacobian $r^2 \sin φ$ for the volume element dv. The bounds depend on the region E, typically expressed in terms of r, θ, and φ.

What is the general form of the integrand (x^2 y^2 z^2) in spherical coordinates?

In spherical coordinates, the integrand becomes: $(r \sin φ \cos θ)^2$ $(r \sin φ \sin θ)^2$ $(r \cos φ)^2$ = $r^6 (\sin φ)^4 (\cos θ)^2 (\sin θ)^2 (\cos φ)^2$.

How do you determine the limits of integration in spherical coordinates for a specific region E?

The limits depend on the shape of region E. Typically, r varies from 0 to some maximum radius, θ from 0 to 2π, and φ from 0 to π, but these can be restricted based on the boundaries of E. For example, if E is a sphere of radius R, then $r \in [0, R]$, $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$.

What is the volume element dv in spherical coordinates?

The volume element in spherical coordinates is dv = r^2 sin φ dr dφ dθ.

How does symmetry help in simplifying the triple integral of x^2 y^2 z^2 over symmetric regions?

Symmetry can simplify the integral by eliminating terms that integrate to zero over symmetric limits, and by recognizing that certain integrals involving odd functions over symmetric bounds are zero, reducing the computational effort.

What are common challenges when evaluating triple integrals in spherical coordinates?

Common challenges include correctly transforming the integrand, setting accurate limits for the region E, and managing the complexity of the integrand involving multiple trigonometric functions, which may require using symmetry or standard integral formulas.

How do you evaluate the integral of (x^2 y^2 z^2) dv over a spherical region E?

Transform the integrand into spherical coordinates, express dv accordingly, determine the limits based on E, and then perform the triple integral by integrating with respect to r, φ, and θ, often using standard integral tables or computational tools for the trigonometric parts.

Additional Resources

Use Spherical Coordinates To Evaluate The Triple Integral \(\iiint_E (x^2 y^2 z^2) \, dv\), where \(E\) Is

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Introduction

Mathematics, especially multivariable calculus, offers powerful tools to evaluate complex integrals over three-dimensional regions. Among these tools, spherical coordinates stand out for their elegance and efficiency when dealing with regions possessing spherical symmetry. This article explores the process of transforming a triple integral involving the integrand $x^2 y^2 z^2$ into spherical coordinates, providing a comprehensive analysis of the methodology, calculations, and implications.

Understanding the Integral and Region E

The Nature of the Integrand

The integrand $x^2 y^2 z^2$ is a polynomial function symmetric with respect to each variable. Its structure suggests that the region of integration, E, could be a sphere or a region bounded by spherical surfaces, given that spherical coordinates are most suitable for such regions.

Defining the Region E

While the original prompt does not specify the exact bounds of the region E, typical problems involve regions such as:

- The entire sphere $r \leq R$,
- Spherical caps,
- Spherical shells,
- Regions bounded by specific spheres or cones.

For the purposes of this analysis, we will assume E is the solid sphere of radius R, i.e.,

$$
E = \{ (x,y,z) \mid x^2 + y^2 + z^2 \leq R^2 \}.
$$

This choice aligns with the use of spherical coordinates and allows for a detailed demonstration.

Transition from Cartesian to Spherical Coordinates

Spherical Coordinate System Overview

Spherical coordinates (r, θ, ϕ) are related to Cartesian coordinates (x, y, z) via:

$$
\begin{cases}
x = r \sin \phi \cos \theta, \\
y = r \sin \phi \sin \theta, \\
z = r \cos \phi,
$$

\end{cases}
\]

where:

- \(r \geq 0\) is the radius,
- \(\theta \in [0, 2\pi]\) is the azimuthal angle in the \(xy\)-plane,
- \(\phi \in [0, \pi]\) is the polar angle measured from the positive \(z\)-axis.

Jacobian Determinant for the Transformation

The volume element \(dv\) in Cartesian coordinates transforms to spherical coordinates as:

\[
dv = r^2 \sin \phi \, dr \, d\phi \, d\theta.
\]

This Jacobian accounts for the change of variables and ensures the integral's correctness.

Expressing the Integrand in Spherical Coordinates

The integrand \(x^2 y^2 z^2\) is expressed in spherical coordinates as follows:

\[
x^2 y^2 z^2 = (r \sin \phi \cos \theta)^2 \times (r \sin \phi \sin \theta)^2 \times (r \cos \phi)^2.
\]

Simplifying,

\[
x^2 y^2 z^2 = r^6 \sin^4 \phi \cos^2 \theta \sin^2 \theta \cos^2 \phi.
\]

The multiple trigonometric terms can be further simplified using identities:

- \(\sin^2 \theta \cos^2 \theta = \frac{1}{4} \sin^2 2\theta\),
- \(\sin^4 \phi = (\sin^2 \phi)^2\),
- \(\cos^2 \phi = 1 - \sin^2 \phi\).

However, for integration purposes, expressing the integrand in terms of these known identities facilitates easier integration, especially over symmetric bounds.

Setting Up the Integral in Spherical Coordinates

The General Form

Given the assumptions, the triple integral becomes:

$$
\iiint_E x^2 y^2 z^2 \, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^6 \sin^4 \phi \cos^2 \theta \sin^2 \theta \cos^2 \phi \times r^2 \sin \phi \, dr \, d\phi \, d\theta.
$$

Simplify the integrand:

$$
r^6 \times r^2 = r^8,
$$
and the angular part:

$$
\sin^4 \phi \times \sin \phi = \sin^5 \phi,
$$
so the integrand simplifies to:

$$
r^8 \sin^5 \phi \times \cos^2 \phi \times \cos^2 \theta \times \sin^2 \theta.
$$

Therefore, the integral is:

$$
\iiint_E x^2 y^2 z^2 \, dv = \int_0^{2\pi} \int_0^{\pi} \int_0^{R} r^8 \sin^5 \phi \cos^2 \phi \cos^2 \theta \sin^2 \theta \, dr \, d\phi \, d\theta.
$$

Evaluating the Integral: Step-by-Step Approach

The triple integral separates into three integrals due to the independence of the variables:

$$
\iiint_E = \left(\int_0^R r^8 \, dr \right) \times \left(\int_0^\pi \sin^5 \phi \cos^2 \phi \, d\phi \right) \times \left(\int_0^{2\pi} \cos^2 \theta \sin^2 \theta \, d\theta \right).
$$

1. Radial Integral

\[

\int_0^R r^8 \, dr = \frac{R^{9}}{9}.
\]

2. Angular Integral in \(\phi\)

\[
I_\phi = \int_0^\pi \sin^5 \phi \cos^2 \phi \, d\phi.
\]

Use substitution \(u = \cos \phi\), \(du = - \sin \phi \, d\phi\). When
\(\phi = 0\), \(u=1\); when \(\phi=\pi\), \(u=-1\). The integral becomes:

\[
I_\phi = \int_{u=1}^{-1} \sin^4 \phi \, u^2 \times (- du).
\]

Note that \(\sin \phi = \sqrt{1 - u^2}\), so:

\[
I_\phi = \int_{-1}^{1} (1 - u^2)^2 u^2 \, du.
\]

Expanding numerator:

\[
(1 - u^2)^2 u^2 = u^2 (1 - 2 u^2 + u^4) = u^2 - 2 u^4 + u^6.
\]

Thus,

\[
I_\phi = \int_{-1}^{1} u^2 \, du - 2 \int_{-1}^{1} u^4 \, du + \int_{-1}^{1}
u^6 \, du.
\]

Since the integrand is even, integrate over \([0,1]\) and double:

\[
I_\phi = 2 \left[\int_0^1 u^2 \, du - 2 \int_0^1 u^4 \, du + \int_0^1 u^6 \,
du \right].
\]

Compute each:

- \(\int_0^1 u^2 \, du = \frac{1}{3}\),
- \(\int_0^1 u^4 \, du = \frac{1}{5}\),
- \(\int_0^1 u^6 \, du = \frac{1}{7}\).

Substitute:

\[

I_\phi = 2 \left(\frac{1}{3} - 2 \times \frac{1}{5} + \frac{1}{7} \right) =
2 \left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7} \right).
\]

Find common denominator \(105\):

\[
\frac{35}{105} - \frac{42}{105} + \frac{15}{105} = \frac{35 - 42 + 15}{105} =
\frac{8}{105}.
\]

Multiply by 2:

\[
I_\phi = 2 \times \frac{8}{105} = \frac{16}{105}.
\]

3. Angular Integral in \(\theta\)

\[
I_\theta = \int_0^{2\pi} \cos^2 \theta \sin^2 \theta \, d\theta.
\]

Use the identity:

\[
\sin 2\theta = 2 \sin \theta \cos \theta,
\]
which implies:

\[
\sin^2 2\theta = 4 \sin^2 \theta

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