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10. (22 Points) Use The Laplace Transform To Solve The Given IVP. Y (0) = 0, Y"+y' - 2y = 3 Cos (3t)

Solving differential equations is a fundamental aspect of engineering, physics, and applied mathematics. Among the various techniques available, the Laplace Transform stands out as a powerful tool for handling linear ordinary differential equations (ODEs), especially those with initial conditions and forcing functions involving trigonometric or exponential functions. In this article, we will explore how to solve the initial value problem (IVP):

$Y'' + Y' - 2Y = 3\cos(3t)$, with initial condition $Y(0) = 0$,

using the Laplace Transform method. This approach transforms the differential equation from the time domain into the algebraic domain, simplifying the process of finding the particular and homogeneous solutions, and ultimately providing a comprehensive solution to the IVP.

Understanding the Problem and Its Components

Before diving into the solution process, it is essential to understand the components of the problem:

- The differential equation: $Y'' + Y' - 2Y = 3\cos(3t)$
- Initial condition: $Y(0) = 0$
- The forcing function: $3\cos(3t)$, a sinusoidal external input

The goal is to find a function $Y(t)$ that satisfies both the differential equation and the initial condition.

Applying the Laplace Transform

The Laplace Transform converts functions from the time domain to the complex frequency domain, defined as:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st}f(t)\, dt,
$$

where s is a complex variable.

Step 1: Take the Laplace Transform of Both Sides

Applying the Laplace Transform to the entire differential equation:

$$
\mathcal{L}\{Y'' + Y' - 2Y\} = \mathcal{L}\{3 \cos(3t)\}
$$

Using linearity:

$$
\mathcal{L}\{Y''\} + \mathcal{L}\{Y'\} - 2 \mathcal{L}\{Y\} = \mathcal{L}\{3 \cos(3t)\}
$$

Recall the Laplace Transform of derivatives:

$$
\mathcal{L}\{Y'\} = sY(s) - Y(0),
$$
$$
\mathcal{L}\{Y''\} = s^2 Y(s) - sY(0) - Y'(0).
$$

Given $Y(0) = 0$, but $Y'(0)$ is not specified, so we denote it as $Y'(0) = y'_0$.

Applying these to the equation:

$$
s^2 Y(s) - s \cdot 0 - y'_0 + s Y(s) - 0 - 2 Y(s) = \mathcal{L}\{3 \cos(3t)\}
$$

Simplify:

$$
s^2 Y(s) - y'_0 + s Y(s) - 2 Y(s) = 3 \mathcal{L}\{\cos(3t)\}
$$

Step 2: Find the Laplace Transform of the Forcing Function

The Laplace Transform of $\cos(3t)$ is:

$$
\mathcal{L}\{\cos(3t)\} = \frac{s}{s^2 + 9}
$$

Therefore,

$$
\mathcal{L}\{3 \cos(3t)\} = 3 \times \frac{s}{s^2 + 9} = \frac{3s}{s^2 + 9}
$$

Formulating the Algebraic Equation in $Y(s)$

Putting everything together:

$$
(s^2 + s - 2) Y(s) - y'_0 = \frac{3s}{s^2 + 9}
$$

Rearranged:

$$
(s^2 + s - 2) Y(s) = y'_0 + \frac{3s}{s^2 + 9}
$$

To proceed, the value of $y'_0 = Y'(0)$ is required. If not specified, standard practice is to assume initial rest, i.e., $Y'(0) = 0$. For the purposes of this solution, we assume $Y'(0) = 0$. If a different initial velocity is given, substitute accordingly.

Thus, the algebraic equation simplifies to:

$$
(s^2 + s - 2) Y(s) = \frac{3s}{s^2 + 9}
$$

and

$$
Y(s) = \frac{3s}{(s^2 + 9)(s^2 + s - 2)}
$$

Partial Fraction Decomposition

To invert $Y(s)$, we need to decompose it into simpler fractions.

Step 1: Factor the quadratic denominator

Factor $s^2 + s - 2$:

$$
s^2 + s - 2 = (s + 2)(s - 1)
$$

Thus,

$$
Y(s) = \frac{3s}{(s^2 + 9)(s + 2)(s - 1)}
$$

Step 2: Set up Partial Fractions

Express $Y(s)$ as:

$$
Y(s) = \frac{A s + B}{s^2 + 9} + \frac{C}{s + 2} + \frac{D}{s - 1}
$$

Multiplying both sides by the denominator:

$$
3s = (A s + B)(s + 2)(s - 1) + C (s^2 + 9)(s - 1) + D (s^2 + 9)(s + 2)
$$

Step 3: Solve for Constants A, B, C, D

This involves plugging in strategic values of s to solve for each constant:

- For $s = -2$:

$$
3(-2) = (A(-2) + B)(-2 + 2)(-2 - 1) + C((-2)^2 + 9)(-2 - 1) + D((-2)^2 + 9)(-2 + 2)
$$

Note that $(-2 + 2) = 0$ and $(-2 + 2) = 0$, so terms involving A, C, D simplify:

$$
-6 = 0 + C(4 + 9)(-3) + 0
$$

$$
-6 = C \times 13 \times (-3) \Rightarrow -6 = -39 C \Rightarrow C = \frac{6}{39} = \frac{2}{13}
$$

- For $s = 1$:

$$
3(1) = (A(1) + B)(1 + 2)(1 - 1) + C(1 + 9)(1 - 1) + D(1 + 9)(1 + 2)
$$

Notice $(1 - 1) = 0$, so the first two terms vanish:

$$
3 = 0 + 0 + D (10)(3) \Rightarrow 3 = 30 D \Rightarrow D = \frac{1}{10}
$$

- For $s = 0$:

$$
0 = (A \times 0 + B)(0 + 2)(0 - 1) + C(0 + 9)(0 - 1) + D(0 + 9)(0 + 2)
$$

Calculate each:

$$
0 = B \times 2 \times (-1) + C \times 9 \times (-1) + D \times 9 \times 2
$$

$$
0 = -2 B - 9 C + 18 D
$$

Substitute $C = \frac{2}{13}$, $D = \frac{1}{10}$:

$$
0 = -2 B - 9 \times \frac{2}{13} + 18 \times \frac{1}{10}
$$
$$
0 = -2 B - \frac{18}{13} + \frac{18}{10}
$$

Express all with common denominator 130:

$$
\frac{18}{13} = \frac{180}{130}, \quad \frac{18}{10} = \frac{234}{130}
$$

So,

$$
0 = -2 B - \frac{180}{130} + \frac{234}{130}
$$
$$
0 = -2 B + \frac{54}{130}
$$

$$
-2 B = - \frac{54}{130} \Rightarrow B = \frac{27}{130}
$$

- For s general, solve for A:

Frequently Asked Questions

What is the initial step in solving the differential equation y'' + y' - 2y = 3 cos(3t) using Laplace transforms?

The initial step is to take the Laplace transform of both sides of the differential equation, applying the initial conditions y(0) = 0 and y'(0) to transform the entire equation into algebraic form.

How do you handle the second derivative y'' in the Laplace domain?

The Laplace transform of y'' is $s^2Y(s)$ - s y(0) - y'(0). Given y(0) = 0 and y'(0) is unknown, but typically zero in initial value problems, it simplifies to $s^2Y(s)$.

What is the Laplace transform of the right-hand side

3 cos(3t)?

The Laplace transform of 3 cos(3t) is 3 (s / (s^2 + 9)).

After transforming the differential equation, how do you solve for Y(s)?

Rearrange the algebraic equation in terms of Y(s), isolating it as Y(s) = [Laplace transform of the RHS] / [Laplace transform of LHS], which involves s, s^2, and coefficients.

What techniques are used to invert Y(s) back to y(t) after solving for Y(s)?

Partial fraction decomposition and known Laplace inverse transforms are used to rewrite Y(s) into terms whose inverse transforms are standard functions, allowing you to find y(t).

What is the significance of the initial conditions in solving the IVP using Laplace transforms?

Initial conditions are essential for transforming derivatives accurately and for solving algebraically for Y(s). They ensure the solution satisfies the given initial value constraints.

How do you interpret the particular solution in this context?

The particular solution corresponds to the steady-state response of the system to the forcing function 3 cos(3t), obtained by inverse Laplace transforming the nonhomogeneous part of Y(s).

What is the general form of the solution after applying the Laplace Transform method to this IVP?

The general solution is the sum of the homogeneous solution (related to the characteristic equation) and the particular solution (derived from the inverse Laplace transform of the nonhomogeneous term), satisfying the initial conditions.

Additional Resources

Laplace Transform: Unlocking the Power to Solve Differential Equations with Ease

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Introduction: The Cornerstone of Differential

Equation Solving

In the realm of engineering and applied mathematics, differential equations serve as fundamental tools to model dynamic systems—be it electrical circuits, mechanical vibrations, or thermal processes. Among the array of methods available for solving these equations, the Laplace Transform stands out as a particularly powerful technique, especially for initial value problems (IVPs). Its ability to convert differential equations from the time domain into algebraic equations in the complex frequency domain simplifies complex calculus operations into straightforward algebraic manipulations.

This article explores the application of the Laplace Transform to solve a specific IVP:

$$
Y'' + Y' - 2Y = 3 \cos(3t), \quad \text{with} \quad Y(0) = 0, \quad Y'(0) = 0.
$$

We'll walk through each step meticulously, ensuring a comprehensive understanding of the process and highlighting why the Laplace Transform remains an essential tool for engineers, mathematicians, and students alike.

Understanding the Differential Equation and Initial Conditions

The Differential Equation

The given second-order linear nonhomogeneous differential equation is:

$$
Y'' + Y' - 2Y = 3 \cos(3t).
$$

This equation models a system where the output $Y(t)$ responds to a cosine forcing function, which could represent a variety of physical phenomena such as forced vibrations or electrical signals.

Initial Conditions

The initial conditions specify the state of the system at $t=0$:

$$
Y(0) = 0, \quad Y'(0) = 0.
$$

These conditions are critical because the Laplace Transform inherently incorporates initial values, making it straightforward to handle such problems.

Step 1: Applying the Laplace Transform

The Power of Transformation

The Laplace Transform $\mathcal{L}$ of a function $f(t)$ is defined as:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_{0}^{\infty} e^{-st}f(t) dt,
$$

which converts functions from the time domain into the complex s-domain. This transformation turns derivatives into algebraic expressions, simplifying differential equations dramatically.

Transforming Each Term

Applying the Laplace Transform to the differential equation involves transforming each term:

- $\mathcal{L}\{Y''\} = s^{2}Y(s) - sY(0) - Y'(0)$
- $\mathcal{L}\{Y'\} = sY(s) - Y(0)$
- $\mathcal{L}\{Y\} = Y(s)$

Given $Y(0) = 0$ and $Y'(0) = 0$, these simplify further.

The right side, $3 \cos(3t)$, transforms using the standard Laplace pair:

$$
\mathcal{L}\{\cos(\omega t)\} = \frac{s}{s^{2} + \omega^{2}},
$$

so:

$$
\mathcal{L}\{3 \cos(3t)\} = 3 \times \frac{s}{s^{2} + 9}.
$$

Step 2: Setting Up the Algebraic Equation

Applying the Laplace Transform to the entire differential equation yields:

$$
s^{2}Y(s) - s \times 0 - 0 + sY(s) - 0 - 2Y(s) = \frac{3s}{s^{2} + 9}.
$$

Simplifying, this becomes:

$$
(s^{2} + s - 2)Y(s) = \frac{3s}{s^{2} + 9}.
$$

Rearranged, the algebraic equation for $Y(s)$ is:

$$
Y(s) = \frac{3s}{(s^{2} + 9)(s^{2} + s - 2)}.
$$

This expression captures the entire solution process in a single algebraic form, ready for partial fraction decomposition.

Step 3: Partial Fraction Decomposition

Factorizing the Denominator

The denominator factors into quadratics and linear factors:

$$
s^{2} + s - 2 = (s + 2)(s - 1),
$$

so the entire denominator is:

$$
(s^{2} + 9)(s + 2)(s - 1).
$$

Decomposition Strategy

Express $Y(s)$ as a sum of simpler fractions:

$$
Y(s) = \frac{3s}{(s^{2} + 9)(s + 2)(s - 1)} = \frac{A s + B}{s^{2} + 9} + \frac{C}{s + 2} + \frac{D}{s - 1}.
$$

This form allows us to find constants A, B, C, D by equating coefficients after clearing denominators.

Solving for Constants

Multiply both sides by the denominator:

$$
3s = (A s + B)(s + 2)(s - 1) + C (s^{2} + 9)(s - 1) + D (s^{2} + 9)(s + 2).
$$

Expanding each term:

- $(A s + B)(s + 2)(s - 1)$
- $C (s^{2} + 9)(s - 1)$
- $D (s^{2} + 9)(s + 2)$

After expansion, equate coefficients of powers of s to solve for A, B, C, D.

Step 4: Calculating Constants and Inverse Laplace Transform

Determining Coefficients

Perform algebraic expansion:

1. Expand $(s + 2)(s - 1) = s^{2} + s - 2$.

2. The first term:

$$
(A s + B)(s^{2} + s - 2) = A s (s^{2} + s - 2) + B (s^{2} + s - 2) = A s^{3} + A s^{2} - 2A s + B s^{2} + B s - 2 B.
$$

3. The second term:

$$
C (s^{2} + 9)(s - 1) = C [s^{3} - s^{2} + 9s - 9].
$$

4. The third term:

$$
D (s^{2} + 9)(s + 2) = D [s^{3} + 2 s^{2} + 9 s + 18].
$$

Adding all:

$$
3 s = \left[A s^{3} + A s^{2} - 2A s + B s^{2} + B s - 2 B \right] + \left[C s^{3} - C s^{2} + 9 C s - 9 C \right] + \left[D s^{3} + 2 D s^{2} + 9 D s + 18 D \right].
$$

Group like terms:

- s^{3}:

$$
A + C + D
$$

- s^{2}:

$$
A + B - C + 2 D
$$

- s:

$$
-2A + B + 9 C + 9 D
$$

- Constant term:

$$
-2 B - 9 C + 18 D
$$

Set equal to the left side:

$$
3 s + 0.
$$

Matching coefficients:

- For s^{3}:

$$
A + C + D = 0,
$$

- For s^{2}:

$$
A + B - C + 2 D = 0,
$$

- For s:

$$
-2A + B + 9 C + 9 D = 3,
$$

- For constants:

$$
-2 B - 9 C + 18 D = 0.
$$

This system can be solved sequentially for A, B, C, D.

Step 5: Solving for Constants

Let’s proceed step by step:

1. From the first equation:

$$
A = -C - D.
$$

2. Substitute into the second:

$$
(-C - D) + B - C + 2 D = 0,
$$

which simplifies to:

$$
B - 2 C + D = 0,
$$

or

$$
B = 2 C - D.
$$

3. From the third:

$$
-2(-C - D) + B + 9 C + 9 D = 3,
$$

which simplifies to:

$$
2 C
$$

10 22 Points Use The Laplace Transform To Solve The Given Ivp Y 0 0 Y Y 2y 3 Cos 3t

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