3.2.1. If The Random Variable X Has A Poisson Distribution Such That P(X = 1) = P(X = 2), Find P(X =

3.2.1. If The Random Variable X Has A Poisson Distribution Such That P(X = 1) = P(X = 2), Find P(X = and explore the problem comprehensively, including the properties of the Poisson distribution, the process of solving for the probability, and related concepts in probability theory.

Understanding the Poisson Distribution

The Poisson distribution is a discrete probability distribution that models the number of events occurring within a fixed interval of time or space, given that these events happen independently and at a constant average rate. It is widely used in fields such as queuing theory, telecommunications, biology, and reliability engineering.

Definition and Probability Mass Function

The probability mass function (PMF) of a Poisson-distributed random variable X is given by:

$$
P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}
$$

where:
- k is a non-negative integer (0, 1, 2, ...),
- $\lambda > 0$ is the average rate (mean number of events in the interval).

The parameter λ essentially captures the expected number of occurrences within the interval, and the distribution is characterized entirely by λ.

Key Properties of the Poisson Distribution

- Mean and Variance: Both are equal to λ.
- Memoryless Property: Unlike the exponential distribution (which models waiting times), the Poisson distribution models counts and is not memoryless.
- Additivity: The sum of independent Poisson random variables is also Poisson-distributed, with the parameter being the sum of individual

parameters.

Problem Statement and Approach

Given:
$$
P(X = 1) = P(X = 2)
$$
for a Poisson random variable X. Our goal is to find $P(X = k)$ for some specific k based on this condition.

The key steps involve:
1. Expressing $P(X=1)$ and $P(X=2)$ in terms of λ.
2. Setting these probabilities equal to each other.
3. Solving for λ.
4. Calculating the desired probability $P(X = k)$.

Deriving the Parameter λ

Expressing the Probabilities

Using the Poisson PMF:
$$
P(X = 1) = \frac{\lambda^1 e^{-\lambda}}{1!} = \lambda e^{-\lambda}
$$
$$
P(X = 2) = \frac{\lambda^2 e^{-\lambda}}{2!} = \frac{\lambda^2 e^{-\lambda}}{2}
$$

Setting the Probabilities Equal

Given:
$$
P(X=1) = P(X=2)
$$
substitute the expressions:
$$
\lambda e^{-\lambda} = \frac{\lambda^2 e^{-\lambda}}{2}
$$

Since $e^{-\lambda} \neq 0$, we can divide both sides by $e^{-\lambda}$:
$$

\lambda = \frac{\lambda^2}{2}
\]

This simplifies to:
\[
2 \lambda = \lambda^2
\]

or:
\[
\lambda^2 - 2 \lambda = 0
\]

Factor out \(\lambda\):
\[
\lambda (\lambda - 2) = 0
\]

Therefore:
\[
\lambda = 0 \quad \text{or} \quad \lambda = 2
\]

Since \(\lambda\) (the average rate) must be positive, discard \(\lambda=0\).
Thus:
\[
\boxed{\lambda=2}
\]

Calculating the Desired Probability \(P(X=k)\)

With the parameter \(\lambda=2\), the Poisson distribution simplifies to:
\[
P(X = k) = \frac{2^k e^{-2}}{k!}
\]

Depending on the specific \(k\) you wish to evaluate, substitute accordingly.

Example Calculations

- For \(k=0\):
\[
P(X=0) = \frac{2^0 e^{-2}}{0!} = e^{-2} \approx 0.1353
\]
- For \(k=1\):
\[
P(X=1) = \frac{2^1 e^{-2}}{1!} = 2 e^{-2} \approx 0.2707
\]

- For $k=2$:

$$
P(X=2) = \frac{2^2 e^{-2}}{2!} = \frac{4 e^{-2}}{2} = 2 e^{-2} \approx 0.2707
$$

Notice that this aligns with the initial condition, as $P(X=1) = P(X=2) \approx 0.2707$.

Additional Considerations and Applications

Understanding the condition where $P(X=1) = P(X=2)$ helps in various practical scenarios, such as:

- Model calibration: Adjusting the rate parameter λ to fit observed data.
- Probability computations: Calculating specific event probabilities once λ is known.
- Risk assessment: Estimating the likelihood of certain counts in processes modeled by Poisson processes.

Related Concepts in Probability and Statistics

- Poisson Approximation: The Poisson distribution often approximates binomial distributions for large n and small p where np is moderate.
- Poisson Process: A model of events occurring randomly over time or space, with the number of events in disjoint intervals being independent.
- Parameter Estimation: Methods such as Maximum Likelihood Estimation (MLE) are used to estimate λ from data.

Summary

In this problem, we demonstrated how to derive the Poisson distribution parameter λ from a probability equality condition. The key steps involved:

- Expressing probabilities using the PMF.
- Equating the probabilities and simplifying.
- Solving for λ.

The solution revealed that $\lambda=2$ satisfies the condition $P(X=1)=P(X=2)$. Once the parameter is determined, calculating any specific probability becomes straightforward using the Poisson formula.

This process underscores the power of the Poisson distribution in modeling count data and highlights the importance of understanding probability relationships to infer underlying parameters. Whether applied in quality control, telecommunications, or biological research, such analytical methods form the backbone of statistical inference in discrete event modeling.

Conclusion

The problem of finding the probability \(P(X=k)\) given the condition \(P(X=1) = P(X=2)\) showcases the utility of the Poisson distribution’s properties and the elegance of algebraic solutions in probability theory. By understanding the distribution's form and manipulating the probability equations, we can uncover key parameters and make informed predictions about the process being modeled.

Frequently Asked Questions

Given that a Poisson random variable X satisfies P(X=1) = P(X=2), how can we find P(X=3)?

First, express P(X=1) and P(X=2) in terms of λ: P(X=k) = (e^(-λ) λ^k) / k!. Set P(X=1) = P(X=2): (e^(-λ) λ^1)/1! = (e^(-λ) λ^2)/2!. Simplify to find λ, then substitute into P(X=3) = (e^(-λ) λ^3)/3!.

What is the value of λ if P(X=1) equals P(X=2) in a Poisson distribution?

Setting P(X=1) = P(X=2): (e^(-λ) λ)/1! = (e^(-λ) λ^2)/2!. Cancel e^(-λ): λ = (λ^2)/2, leading to λ = 2.

How do you verify that the probabilities P(X=1) and P(X=2) are equal for the Poisson distribution?

Calculate both probabilities: P(X=1) = (e^(-λ) λ)/1! and P(X=2) = (e^(-λ) λ^2)/2!. Set them equal and solve for λ to verify the condition.

Once λ is found, how do you compute P(X=3) in this scenario?

Insert the value of λ into the Poisson probability formula for P(X=3): P(X=3) = (e^(-λ) λ^3)/3!.

Does the condition P(X=1) = P(X=2) imply any

constraints on the Poisson distribution's mean λ?

Yes, it constrains λ to a specific value; in this case, $\lambda = 2$, as derived from setting the two probabilities equal.

Can the equality P(X=1) = P(X=2) occur for any other λ besides 2 in a Poisson distribution?

No, solving the equation shows that $\lambda=2$ is the unique value satisfying $P(X=1)=P(X=2)$.

What is the significance of the condition P(X=1) = P(X=2) in understanding the Poisson distribution?

It helps identify the specific mean λ at which the probabilities of observing 1 or 2 events are equal, revealing symmetry in the distribution's probabilities.

How would you generalize this problem to find P(X=k) for other values of k given P(X=1)=P(X=2)?

First, find λ from the equality of $P(X=1)$ and $P(X=2)$. Then, use the Poisson formula to compute $P(X=k)$ for any desired k with that λ.

What is the final probability P(X=3) after determining λ from the given condition?

With $\lambda=2$, $P(X=3) = (e^{-2} 2^3)/3! = (e^{-2} 8)/6 \approx 0.1804$.

Additional Resources

Poisson Distribution is a foundational concept in probability theory and statistics, frequently employed to model count-based phenomena where events occur independently within a fixed interval or space. Its versatility makes it invaluable across diverse fields—from telecommunications and epidemiology to finance and natural sciences. This article delves into a specific problem involving the Poisson distribution: determining the probability \(P(X = k) \) given certain conditions, specifically when the probability that \(X \) equals 1 is equal to the probability that \(X \) equals 2. Through meticulous analysis, detailed explanations, and step-by-step derivations, we will explore how to solve this problem and understand its implications.

Understanding the Poisson Distribution

Definition and Probability Mass Function

The Poisson distribution models the probability of a given number of events X occurring in a fixed interval or space, assuming these events occur independently and at a constant average rate λ (the mean number of occurrences). The probability mass function (pmf) of a Poisson-distributed random variable X is given by:

$$
P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}
$$

where:
- k is the number of occurrences (a non-negative integer),
- $\lambda > 0$ is the average rate of occurrence,
- e is Euler's number, approximately 2.71828.

This formula encapsulates how the probability decreases factorially as k increases, with the mean λ governing the distribution's shape and scale.

Key Properties of the Poisson Distribution

- Mean and Variance: Both are equal to λ, i.e., $E[X] = \lambda$ and $Var(X) = \lambda$.
- Memoryless: The Poisson process has no memory; the probability of an event occurring in the future is independent of past events.
- Sum of Poisson Variables: The sum of independent Poisson variables is also Poisson-distributed, with parameter equal to the sum of their individual parameters.

Real-World Applications

- Modeling the number of emails received per hour.
- Counting radioactive decay events in a fixed time.
- Estimating the number of customer arrivals at a service center.
- Analyzing the occurrence of natural events like earthquakes or lightning strikes.

Problem Setup and Context

The problem at hand states:

> "If the random variable X has a Poisson distribution such that $P(X=1) = P(X=2)$, find $P(X= \dots)$."

This formulation indicates that the probabilities of X taking the values 1 and 2 are equal, which provides a crucial equation to determine the parameter λ. Once λ is known, one can find various probabilities $P(X=k)$ for any k.

Restating the problem:

- $X \sim \text{Poisson}(\lambda)$
- $P(X=1) = P(X=2)$

Our goal: Find $P(X=k)$ for some k, or perhaps the general probability distribution. Typically, in such problems, the primary step is to determine λ based on the given condition, then derive the probability of interest.

Deriving the Parameter λ

Expressing the Given Condition

Given the pmf:

$$
P(X=k) = \frac{\lambda^k e^{-\lambda}}{k!}
$$

the condition $P(X=1) = P(X=2)$ becomes:

$$
\frac{\lambda^1 e^{-\lambda}}{1!} = \frac{\lambda^2 e^{-\lambda}}{2!}
$$

Note that $e^{-\lambda}$ appears on both sides and can be canceled out:

$$
\frac{\lambda}{1} = \frac{\lambda^2}{2}
$$

which simplifies to:

$$
\lambda = \frac{\lambda^2}{2}
$$

Solving for λ

Rearranging:

$$
2 \lambda = \lambda^2
$$

which leads to:

$$
\lambda^2 - 2 \lambda = 0
$$

$$
\lambda (\lambda - 2) = 0
$$

Since $\lambda > 0$ (the average rate in a Poisson distribution must be positive), the solution is:

$$
\boxed{
\lambda = 2
}
$$

This critical step reveals that the mean number of events, λ, must be 2 for the probabilities $P(X=1)$ and $P(X=2)$ to be equal.

Calculating Specific Probabilities

With $\lambda = 2$, the entire probability distribution of X is now defined explicitly:

$$
P(X=k) = \frac{2^k e^{-2}}{k!}
$$

for all non-negative integers k.

Probability that $X = 1$ and $X = 2$

- $P(X=1)$:

$$
P(X=1) = \frac{2^1 e^{-2}}{1!} = 2 e^{-2}
$$

- $P(X=2)$:

$$
P(X=2) = \frac{2^2 e^{-2}}{2!} = \frac{4 e^{-2}}{2} = 2 e^{-2}
$$

which confirms the initial condition $P(X=1) = P(X=2)$.

Probability that $X = 0$

$$
P(X=0) = \frac{2^0 e^{-2}}{0!} = e^{-2}
$$

Probability that $X = 3$

$$
P(X=3) = \frac{2^3 e^{-2}}{3!} = \frac{8 e^{-2}}{6} = \frac{4}{3} e^{-2}
$$

General Probabilities and Distribution Characteristics

Knowing $\lambda = 2$, the entire distribution is specified. For any integer $k \geq 0$:

$$
\boxed{
P(X=k) = \frac{2^k e^{-2}}{k!}
}
$$

This allows us to compute probabilities with ease and analyze various features of the distribution.

Key aspects:

- The distribution is centered around the mean $\lambda = 2$.
- The probabilities decrease factorially for larger k, but owing to the small mean, the distribution is heavily weighted towards small k.
- The probability of observing zero events:

$$
P(X=0) = e^{-2} \approx 0.1353
$$

- The probability of observing exactly one event:

$$
P(X=1) = 2 e^{-2} \approx 0.2707
$$

- The probability of observing exactly two events:

$$
P(X=2) = 2 e^{-2} \approx 0.2707
$$

- The probability of observing three or more events diminishes rapidly.

Implications and Applications of the Result

The determination that $\lambda = 2$ based on the equality $P(X=1) = P(X=2)$ provides valuable insights into the behavior of the modeled process.

Practical Interpretation

Suppose this Poisson process models, for example, the number of emails received per hour at a company. The derived mean rate of 2 emails per hour aligns with observed data where the probabilities of receiving exactly 1 or 2 emails are equal. This symmetry indicates that, in this context, the process naturally balances the likelihood of observing one and two events, perhaps reflecting operational constraints or underlying patterns.

Broader Applications

- Quality Control: Understanding when the probabilities for different counts are equal can help identify specific thresholds or thresholds for intervention.
- Risk Assessment: In fields like insurance, knowing the mean rate helps evaluate the likelihood of certain counts of claims or incidents.

- Natural Phenomena: In environmental sciences, the model might relate to the expected number of natural events such as earthquakes or lightning strikes per unit time.

Analytical Significance

From a theoretical standpoint, the condition $P(X=1) = P(X=2)$ is a particular case that constrains the distribution's parameter. Such conditions are often used in statistical inference to estimate parameters when limited information is available, or to validate models against observed data.

Conclusion

The problem of finding the probability $P(X=k)$ for a Poisson-distributed random variable X, given the condition $P(X=1) = P(X=2)$, reveals the elegant interplay between the distribution's parameters and its probabilities. By setting $P(X=1) = P(X=2)$, we deduced that the mean rate λ must be

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