38. Consider The Solid Region That Lies Under The Surface Z = X Vy And Above The Rectangle R= [0, 2]

Understanding the Solid Region Under the Surface \(Z = XY \) Above the Rectangle \(R = [0, 2] \)

38. Consider The Solid Region That Lies Under The Surface \(Z = XY \) And Above The Rectangle \(R = [0, 2] \). This problem involves analyzing a three-dimensional region bounded by a surface and a rectangular base. Such regions are fundamental in multivariable calculus, especially in setting up and evaluating triple integrals for volume calculations, moments, and other physical or geometric quantities. To comprehend this region thoroughly, we need to explore the geometric setup, the limits of integration, and the methods to evaluate integrals over this region.

Geometric Interpretation of the Surface and Region

The Surface \(Z = XY \)

The surface described by \(Z = XY \) is a hyperbolic paraboloid, a common saddle-shaped surface in three-dimensional space. Its key characteristics include:

- It passes through the origin \((0, 0, 0) \).
- The surface is symmetric with respect to the axes, with positive \(Z \) values where \(X \) and \(Y \) have the same sign, and negative \(Z \) where they have opposite signs.
- The surface exhibits saddle behavior, curving upward in one direction and downward in the perpendicular direction.

The Base Rectangle \(R = [0, 2] \times [0, 2] \)

The rectangle \(R \) in the \(XY \)-plane spans from \(X = 0 \) to \(X = 2 \) and \(Y = 0 \) to \(Y = 2 \). Visualizing this rectangle is crucial:

- It occupies the first quadrant, where both \(X \) and \(Y \) are non-negative.
- Within this domain, \(Z = XY \) is always non-negative, since the product of two non-negative numbers is non-negative.

Defining the Solid Region

Boundaries of the Region

The solid region is bounded below by the rectangle \(R \) in the \(XY \)-plane and above by the surface \(Z = XY \). Specifically:

1. Base boundary: The 2D rectangle \(R \) in the \(XY \)-plane.

2. Upper boundary: The surface \(Z = XY \), which extends upward from the \(XY \)-plane.

Visualizing the Region

Imagine the rectangle \(R \) lying flat on the \(XY \)-plane. Above each point \((x, y) \) in \(R \), the surface extends upward to the height \(Z = XY \). The three-dimensional volume enclosed between the rectangle and the surface forms the solid region of interest.

Mathematical Description of the Region

Limits of Integration

To set up an integral over this volume, we need to define the limits for \(X \), \(Y \), and \(Z \). Since the base is fixed and the surface is explicitly given, the natural choice is:

- \(x \) varies from 0 to 2.

- For each fixed \(x \), \(y \) varies from 0 to 2.

- \(Z \) varies from 0 (the \(XY \)-plane) up to \(XY \).

Describing the Volume Element

The differential volume element in Cartesian coordinates is \(dV = dx\,dy\,dz \). To integrate over the solid, we can set up the integral as:

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V = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{XY} dz\,dy\,dx
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Evaluating the Volume of the Solid Region

Setting Up the Triple Integral

Given the bounds, the volume V can be computed as:

$$V = \int_{0}^{2} \int_{0}^{2} \int_{0}^{xy} dz\,dy\,dx$$

Integrating with respect to z first:

$$V = \int_{0}^{2} \int_{0}^{2} xy\,dy\,dx$$

Reducing to a Double Integral

The problem simplifies to calculating a double integral:

$$V = \int_{0}^{2} \left(\int_{0}^{2} xy\,dy \right) dx$$

Computing the Inner Integral

For a fixed x, evaluate:

$$\int_{0}^{2} xy\,dy = x \int_{0}^{2} y\,dy = x \left[\frac{y^{2}}{2} \right]_0^2 = x \times \frac{4}{2} = 2x$$

Computing the Outer Integral

Now, integrate with respect to x:

$$V = \int_{0}^{2} 2x\,dx = 2 \int_{0}^{2} x\,dx = 2 \left[\frac{x^{2}}{2} \right]_0^2 = 2 \times \frac{4}{2} = 4$$

Interpretation of the Result

The volume of the solid region bounded above by $Z = XY$ over the rectangle $R = [0, 2] \times [0, 2]$ is 4 cubic units. This calculation provides a quantitative measure of the space enclosed between the surface and the base in the specified domain.

Applications and Further Implications

Applications in Physics and Engineering

- Calculating mass or charge distribution over a region with density proportional to XY.
- Determining the volume for fluid or material flow in a bounded region.

Extensions to Surface Integrals and Moments

Beyond volume calculation, similar setups allow for computing surface integrals, moments of inertia, and center of mass by integrating different functions over the same region.

Visual Aids and Graphical Representation

Utilizing 3D graphing tools can enhance understanding of the region. Visualizing the surface $Z = XY$ over the rectangle reveals the saddle shape and how the volume accumulates as x and y vary from 0 to 2.

- Plot the surface $Z = XY$ over the domain.
- Shade the region under the surface within the rectangle.
- Observe how the height Z increases with x and y.

Summary and Key Takeaways

- The solid region is bounded below by the rectangle $R = [0, 2] \times [0, 2]$ and above by the surface $Z = XY$.
- Its volume can be computed via a straightforward double integral after integrating out Z.
- The calculated volume is 4 cubic units, illustrating the application of multivariable calculus techniques.
- This analysis serves as a foundation for more complex volume and surface integral calculations in applied mathematics and physics.

Conclusion

Analyzing the solid region under the surface \(Z = XY \) above the rectangle \(R = [0, 2] \) demonstrates fundamental techniques in multivariable calculus. Setting up the correct limits, understanding the geometric interpretations, and performing the integration systematically allows us to evaluate the volume of complex 3D regions efficiently. Such skills are essential in various scientific and engineering fields where spatial analysis and integral calculus are pivotal.

Frequently Asked Questions

What is the geometric shape of the solid region defined by the surface z = xy and the rectangle R = [0, 2]?

The solid region is a three-dimensional shape lying under the surface $z = xy$ over the rectangular base $R = [0, 2]$ along both x and y axes, forming a volume bounded below by the xy-plane and above by the surface.

How do you set up the double integral to find the volume of the solid region under z = xy over R = [0, 2]?

The volume is given by the double integral $V = \iint_R xy \, dA$, where R is the rectangle $[0, 2] \times [0, 2]$, so $V = \int_0^2 \int_0^2 xy \, dy \, dx$.

What is the value of the volume under the surface z = xy over the rectangle R = [0, 2]?

Calculating the integral, the volume $V = \int_0^2 \int_0^2 xy \, dy \, dx = \int_0^2 [(1/2) x y^2]_0^2 dx = \int_0^2 (1/2) x (4) dx = 2 \int_0^2 x \, dx = 2 [(1/2) x^2]_0^2 = 2 ((1/2) 4) = 2\ 2 = 4$.

Can this problem be extended to find the surface area of the surface z = xy over R? If so, how?

Yes, the surface area can be found using the surface area formula for a graph $z = xy$: $A = \iint_R \sqrt{(1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2)} \, dA$. Here, $\partial z/\partial x = y$ and $\partial z/\partial y = x$, so the integrand becomes $\sqrt{(1 + y^2 + x^2)}$.

What are the partial derivatives of z = xy, and how are they used in surface area calculations?

The partial derivatives are $\partial z/\partial x = y$ and $\partial z/\partial y = x$. They are used in the surface area formula to compute the integrand $\sqrt{(1 + y^2 + x^2)}$, which accounts for the surface's slope at each point.

How does the choice of the rectangle R = [0, 2] influence the

calculation of the volume under z = xy?

The rectangle R defines the bounds of integration for x and y, limiting the region over which the volume is calculated. Its size directly impacts the integral's limits and the resulting volume value.

What are potential applications of calculating the volume under the surface z = xy over a rectangle?

Applications include determining the mass of a surface with density proportional to xy, calculating the amount of material needed for a surface, or modeling physical phenomena like heat distribution over a region.

How can the symmetry of the surface z = xy over the rectangle R = [0, 2] help simplify calculations?

Since the region is in the first quadrant where x, y ≥ 0, and the integrand xy is positive, symmetry can sometimes be exploited in similar problems. However, here, the bounds are straightforward, so direct integration is most efficient. For regions symmetric about axes, symmetry can reduce computation.

Additional Resources

Solid Region Under the Surface Z = XY and Above the Rectangle R = [0, 2] × [0, 2]

When delving into the realm of multivariable calculus, one of the fundamental yet fascinating tasks is analyzing regions and volumes defined by surfaces and boundaries. Among these, the problem of considering the solid region lying under a specific surface and above a particular rectangle is a classic, yet rich, example that offers deep insights into the interplay of geometry, calculus, and spatial reasoning.

In this article, we will thoroughly explore the solid region bounded by the surface \(Z = XY \) and the rectangle \(R = [0, 2] \times [0, 2] \). We will examine its geometric properties, understand the methods to compute its volume, and discuss broader applications and implications of such an analysis.

Understanding the Surface and the Region

The Surface \(Z = XY \)

The surface \(Z = XY \) is a classic example of a saddle-shaped hyperbolic paraboloid. Its defining characteristics include:

- Bilinear nature: The surface is a product of the two variables X and Y.
- Symmetry: It is symmetric with respect to the planes $X = 0$ and $Y = 0$.
- Range: Since $X, Y \in [0, 2]$, the surface extends over the first quadrant portion of the domain, where both X and Y are non-negative.

Visualizing $Z = XY$:

- At the origin $(0, 0)$, the surface meets the plane $Z = 0$.
- Along the axes $X = 0$ or $Y = 0$, the surface also dips to zero.
- The maximum values of Z within the domain occur at $(2, 2)$, where $Z = 4$.

This surface exhibits an increasing trend in both X and Y, reaching a peak at the corner $(2, 2)$, producing a "bowl" shape that rises from the origin.

The Rectangle $R = [0, 2] \times [0, 2]$

This is a square region in the XY-plane, with vertices at:

- $(0, 0)$
- $(2, 0)$
- $(2, 2)$
- $(0, 2)$

This domain is straightforward to describe and provides a manageable area for integration. It encompasses the entire positive quadrant between 0 and 2 in both axes, making it ideal for analyzing the volume under the surface $Z = XY$.

Defining the Solid Region

The solid region S is described as:

- Below the surface $Z = XY$,
- Above the rectangle R.

Mathematically, this can be expressed as:

$$
S = \left\{ (x, y, z) \mid (x, y) \in R, 0 \leq z \leq xy \right\}
$$

This description indicates that for each point (x, y) within the domain R, the vertical extent of the solid stretches from the XY-plane $(z=0)$ up to the surface $Z=XY$.

In essence, the volume of this solid is the integral of the surface $Z=XY$ over the rectangle R:

$$
V = \iint_{R} xy \, dA
$$

where dA is the differential area element in the XY-plane.

Calculating the Volume of the Solid

Setting Up the Double Integral

Given the description, the volume V can be expressed as:

$$
V = \int_{x=0}^{2} \int_{y=0}^{2} xy \, dy \, dx
$$

or equivalently,

$$
V = \int_{y=0}^{2} \int_{x=0}^{2} xy \, dx \, dy
$$

The choice of order of integration is flexible; however, the order should be selected to simplify calculations.

Performing the Integration

Let's evaluate the integral in the order of integrating with respect to y first, then x:

$$
V = \int_{x=0}^{2} \left(\int_{y=0}^{2} xy \, dy \right) dx
$$

Inner integral:

$$
\int_{0}^{2} xy \, dy = x \int_{0}^{2} y \, dy = x \left[\frac{y^2}{2} \right]_0^{2} = x \left(\frac{4}{2} \right) = 2x
$$

Outer integral:

$$
V = \int_{0}^{2} 2x \, dx = 2 \int_{0}^{2} x \, dx = 2 \left[\frac{x^2}{2} \right]_0^{2} = 2 \times \left(\frac{4}{2} \right) = 2 \times 2 = 4
$$

Result:

$$
\boxed{V = 4}
$$

This straightforward calculation reveals that the volume of the solid under the surface $Z = XY$ over the rectangle R is exactly 4 cubic units.

Interpreting the Result and Broader Insights

Geometric Significance

The volume computed represents the "space" between the XY-plane and the surface $Z = XY$ over the square R. Because $Z = XY$ is always non-negative within the domain, the volume is positive and reflects the combined effect of the bilinear surface's shape.

This result also exemplifies how the bilinear nature of the surface influences the volume. The integral of xy over the square is a product of the integrals of x and y, given the separability of the integrand.

Possible Variations and Extensions

The analysis can be extended or modified in several ways:

- Changing the Domain: For example, integrating over $R' = [a, b] \times [c, d]$ with different bounds.
- Using Different Surfaces: Replacing $Z = XY$ with other functions, such as $Z = X^2 + Y^2$ or $Z = \sin(XY)$, to analyze more complex regions.
- Adding Constraints: Considering regions where $Z \geq 0$, or where $Z \leq XY$, to explore volumes under or above certain surfaces.
- Surface Integrals: Extending the analysis to surface areas and flux computations related to the surface $Z = XY$.

Applications and Relevance in Real-World Contexts

Understanding such regions is not merely an academic exercise. These calculations underpin various scientific and engineering disciplines:

- Physics: Computing mass or charge distributions over a region with a density function proportional to xy.
- Engineering: Determining volume or material properties for components shaped by bilinear surfaces.
- Economics: Modeling profit or cost regions where the payoff function is bilinear.
- Environmental Science: Estimating volumes of pollutants or resources distributed over a geographic area with variable intensity.

Conclusion

The exploration of the solid region under the surface $Z=XY$ over the rectangle $R = [0, 2] \times [0, 2]$ epitomizes the elegance of multivariable calculus in bridging geometry and analysis. The straightforward integral setup and evaluation showcase how mathematical tools allow us to quantify and understand complex spatial regions efficiently.

This analysis not only provides a concrete example of volume calculation but also lays a foundation for tackling more intricate problems involving diverse surfaces and domains. Whether in academia or applied sciences, such methods are invaluable for modeling, analysis, and problem-solving in multidimensional contexts.

In sum, the volume of the specified solid is precisely 4 cubic units, a testament to the power of integral calculus in deciphering the geometry of three-dimensional regions defined by simple yet profound functions.

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