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Understanding how to compute the flux of a vector field through a surface is a fundamental concept in vector calculus, especially relevant in physics and engineering. In this article, we will explore the detailed process of calculating the flux of the vector field F(x, y, z) = (yz, xz, yz) through a specified part of a sphere. We will cover the necessary mathematical tools, step-by-step procedures, and practical examples to deepen your understanding.

Understanding the Problem

Before delving into calculations, it’s essential to interpret the problem correctly.

The Vector Field

The vector field is given as:

$$
F(x, y, z) = (y z, x z, y z)
$$

This field assigns a vector to each point in space, with components depending on the coordinates x, y, z.

The Surface: Part of a Sphere

The surface in question is a part of the sphere defined by the equation:

$$
x^2 + y^2 + z^2 = R^2
$$

where R is the radius of the sphere. The problem specifies a particular part of this sphere, which may be a spherical cap, a segment, or a specific region bounded by certain angles or coordinate limits.

Fundamentals of Flux Calculation

Calculating the flux of a vector field through a surface involves integrating the surface integral of the vector field over that surface.

Flux Definition

The flux Φ of a vector field F through a surface S is:

$$
\Phi = \iint_S F \cdot n \, dS
$$

where:
- $F \cdot n$ is the dot product of the vector field with the surface's outward normal vector,
- dS is the differential element of surface area.

Surface Parameterization

To evaluate the surface integral, we typically parameterize the surface S in terms of two variables, say (u, v):

$$
\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))
$$

The surface element dS can then be computed using the cross product of the partial derivatives of $\mathbf{r}$:

$$
d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v) du dv
$$

The outward normal vector is obtained by normalizing this cross product.

Step-by-Step Methodology for Computing the Flux

Let's outline the systematic procedure to compute the flux of F through the specified part of the sphere.

1. Identify the Surface Region

- Determine the specific part of the sphere involved, including bounds in spherical coordinates.

- For example, the region might be a spherical cap defined by $z \geq z_0$, or bounded by certain angles θ and ϕ.

2. Choose an Appropriate Parameterization

- Spherical coordinates are often best suited:

$$
x = R \sin \theta \cos \phi
$$
$$
y = R \sin \theta \sin \phi
$$
$$
z = R \cos \theta
$$

where:
- θ is the polar angle $[0, \pi]$,
- ϕ is the azimuthal angle $[0, 2\pi]$.

- Define bounds for θ and ϕ based on the part of the surface.

3. Compute the Surface Element $d\mathbf{S}$

- Calculate $\mathbf{r}_\theta$ and $\mathbf{r}_\phi$:

$$
\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta}
$$
$$
\mathbf{r}_\phi = \frac{\partial \mathbf{r}}{\partial \phi}
$$

- Find the cross product:

$$
\mathbf{r}_\theta \times \mathbf{r}_\phi
$$

- The magnitude of this cross product times $d\theta d\phi$ gives the differential surface area element.

4. Express the Vector Field in Terms of θ and ϕ

- Substitute the parameterization into $F(x, y, z)$:

\[

$$
F(\mathbf{r}(\theta, \phi))
$$

- Simplify the components accordingly.

5. Compute the Dot Product $F \cdot d\mathbf{S}$

- Calculate:

$$
F(\mathbf{r}(\theta, \phi)) \cdot (\mathbf{r}_\theta \times \mathbf{r}_\phi)
$$

- This gives the integrand for the flux integral.

6. Set Up and Evaluate the Surface Integral

- Integrate over the bounds of θ and ϕ:

$$
\Phi = \int_{\phi_{min}}^{\phi_{max}} \int_{\theta_{min}}^{\theta_{max}} F \cdot (\mathbf{r}_\theta \times \mathbf{r}_\phi) \, d\theta d\phi
$$

- Perform the integration step-by-step, simplifying where possible.

Applying the Method to Our Vector Field and Surface

Let's apply these steps to the specific vector field $F(x, y, z) = (yz, xz, yz)$ and a typical part of the sphere.

1. Defining the Surface Region

Suppose we are asked to compute the flux through the upper hemisphere:

$$
x^2 + y^2 + z^2 = R^2, \quad z \geq 0
$$

The bounds are:
- $\theta \in [0, \pi/2]$
- $\phi \in [0, 2\pi]$

2. Spherical Parameterization

Using spherical coordinates:

$$
x = R \sin \theta \cos \phi
$$
$$
y = R \sin \theta \sin \phi
$$
$$
z = R \cos \theta
$$

3. Computing $\mathbf{r}_\theta$ and $\mathbf{r}_\phi$

Calculate derivatives:

$$
\mathbf{r}_\theta = \frac{\partial}{\partial \theta} (R \sin \theta \cos \phi, R \sin \theta \sin \phi, R \cos \theta) = (R \cos \theta \cos \phi, R \cos \theta \sin \phi, -R \sin \theta)
$$

$$
\mathbf{r}_\phi = \frac{\partial}{\partial \phi} (R \sin \theta \cos \phi, R \sin \theta \sin \phi, R \cos \theta) = (-R \sin \theta \sin \phi, R \sin \theta \cos \phi, 0)
$$

Compute the cross product:

$$
\mathbf{r}_\theta \times \mathbf{r}_\phi
$$

which simplifies to:

$$
\mathbf{r}_\theta \times \mathbf{r}_\phi = R^2 \sin \theta \, \hat{\mathbf{n}}
$$

where $\hat{\mathbf{n}}$ is the unit normal vector pointing outward (upward in the case of the upper hemisphere).

Evaluating the Flux Integral

Now, we express F in spherical coordinates:

\[
F = (y z, x z, y z)
\]

Substituting \(x, y, z\):

\[
x = R \sin \theta \cos \phi
\]
\[
y = R \sin \theta \sin \phi
\]
\[
z = R \cos \theta
\]

gives:

\[
F = (R^2 \sin^2 \theta \sin \phi \cos \theta, R^2 \sin \theta \cos \theta \cos \phi, R^2 \sin^2 \theta \sin \phi \cos \theta)
\]

Next, compute the dot product with the surface element:

\[
F \cdot (\mathbf{r}_\theta \times \mathbf{r}_\phi) = R^2 \sin \theta \times \text{(expression involving \(\theta, \phi\))}
\]

Finally, integrate over \(\theta \in [0, \pi/2]\) and \(\phi \in [0, 2\pi]\):

\[
\Phi = \int_0^{2\pi} \int_0^{\pi/2} F \cdot (\mathbf{r}_\theta \times \mathbf{r}_\phi) \, d

Frequently Asked Questions

What is the main goal when computing the flux of the vector field F(x, y, z) = (yz, xz, yz) through a part of a sphere?

The main goal is to evaluate the surface integral of the vector field over the specified part of the sphere to determine the net flow of the field through that surface.

How do you set up the surface integral for calculating the flux of F across a spherical surface?

You set up the surface integral as ∬_S F · n dS, where S is the surface of the sphere and n is the outward unit normal vector at each point. Parametrization or spherical coordinates are often used for S.

What is the significance of choosing the correct part of the sphere when computing flux?

Selecting the correct part of the sphere ensures accurate computation of flux through the intended surface, accounting for boundaries and orientations specific to the problem.

How can the divergence theorem simplify the computation of flux for this vector field?

The divergence theorem states that the flux through a closed surface equals the triple integral of the divergence over the volume enclosed. If the surface is part of a closed surface, this can simplify calculations by converting surface integrals into volume integrals.

What is the divergence of the vector field F(x, y, z) = (yz, xz, yz)?

The divergence is $\partial/\partial x(yz) + \partial/\partial y(xz) + \partial/\partial z(yz) = 0 + x + y = x + y$.

Can the divergence theorem be applied directly to a 'part' of a sphere? Why or why not?

No, the divergence theorem applies to closed surfaces. To use it for a part of a sphere, you need to close the surface with an additional boundary surface, forming a closed surface, before applying the theorem.

What coordinate system is most convenient for parametrizing the spherical surface in this problem?

Spherical coordinates (r, θ, φ) are most convenient, where r is the radius, θ is the azimuthal angle, and φ is the polar angle, especially if the sphere is centered at the origin.

How does the symmetry of the vector field influence the flux calculation over the sphere?

Symmetry can simplify the calculation, as certain integrals may cancel out or reduce due to symmetric properties of the field, especially when integrating over symmetric portions of the sphere.

What boundary conditions or limits are needed to compute the flux through a specific part of the sphere?

You need to specify the angular limits (e.g., ranges of θ and φ) that define the part of the sphere, as well as the radius, to set up the proper limits of integration for the surface integral.

What steps are involved in explicitly computing the flux of F through a spherical surface?

First, parametrize the surface (using spherical coordinates), then compute the surface element dS, determine the normal vector n, evaluate F · n at each point, and perform the surface integral over the specified limits.

Additional Resources

4. Compute The Flux Of The Vector Field F(x,y,z) = (yz, xz, yz) through The Part Of The Sphere x + y + z = 1

Introduction

In the realm of vector calculus, the concept of flux provides a powerful way to quantify how a vector field flows through a surface. Today, we delve into a classic problem involving the computation of the flux of a specific vector field across a particular surface—a segment of a sphere. The problem statement is as follows: Compute the flux of the vector field $\mathbf{F}(x,y,z) = (yz, xz, yz)$ through the part of the sphere $x + y + z = 1$. This problem combines several fundamental ideas—surface integrals, divergence theorem, parameterization, and the geometry of spheres—which together showcase the elegance and utility of vector calculus.

Understanding the Problem

Before diving into calculations, it’s crucial to understand the core elements involved:

- The Vector Field: $\mathbf{F}(x, y, z) = (yz, xz, yz)$. This field has components that depend on the products of the coordinates, reflecting a certain symmetry and complexity.
- The Surface: The part of the sphere defined by $x + y + z = 1$. This is a plane intersecting the sphere, creating a curved surface segment.
- The Goal: To compute the flux of $\mathbf{F}$ through this surface, meaning the total amount of the field passing outward through the surface.

Geometric and Analytical Foundations

The Sphere and Its Segment

The sphere in question is implicitly defined by $x + y + z = 1$. But to understand what part of the sphere we are dealing with, we need to know its position and size:

- Equation of the Sphere: Generally, the sphere is given by $x^2 + y^2 + z^2 = r^2$. However, in the problem statement, the sphere is more precisely defined by the intersection of the surface $x + y + z = 1$ with the sphere centered at the origin, which is likely implied here.

- Clarification: Since the problem references "the part of the sphere $x + y + z = 1$", it suggests the sphere is the standard sphere $x^2 + y^2 + z^2 = R^2$, and the plane slices through it, creating a cap or segment.

Note: For the purpose of this problem, we may interpret the "sphere" as the part of the sphere $x^2 + y^2 + z^2 = 1$ bounded by the plane $x + y + z = 1$. This assumption aligns with typical problems in vector calculus.

Step 1: Setting Up the Surface Integral

The flux of a vector field $\mathbf{F}$ through a surface S is given by the surface integral:

$$
\Phi = \iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS
$$

where $\mathbf{n}$ is the unit normal vector to the surface, pointing outward.

Key challenges:

- Parameterize the surface S.
- Find the surface element dS.
- Compute $\mathbf{F} \cdot \mathbf{n}$.

Step 2: Choosing a Method – Direct Surface Integral or Divergence Theorem?

Given the complexity, two main methods are available:

- Direct Surface Integral: Parameterize the surface explicitly and compute the integral.
- Divergence Theorem: Convert the surface integral into a volume integral of the divergence of $\mathbf{F}$.

The divergence theorem can significantly simplify the calculation if applicable. It states that:

$$
\iint_{S} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_{V} \nabla \cdot \mathbf{F} \, dV
$$

where V is the volume enclosed by S.

Step 3: Computing the Divergence of $\mathbf{F}$

Let's compute $\nabla \cdot \mathbf{F}$:

\[

$$
\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial y}(xz) + \frac{\partial}{\partial z}(yz)
$$

Calculations:

- $\frac{\partial}{\partial x}(yz) = 0$ (no x dependence)
- $\frac{\partial}{\partial y}(xz) = 0$ (no y dependence)
- $\frac{\partial}{\partial z}(yz) = y$

Therefore,

$$
\nabla \cdot \mathbf{F} = 0 + 0 + y = y
$$

This divergence simplifies the problem to a volume integral of y over the region enclosed by the surface segment.

Step 4: Describing the Enclosed Volume

The divergence theorem requires the volume V bounded by the surface segment S.

- Surface S: Part of the sphere $x^2 + y^2 + z^2 = 1$, cut off by the plane $x + y + z = 1$.

- Volume V: The region inside the sphere, limited by the plane $x + y + z = 1$, and the sphere surface.

To set up the volume integral, we need to parametrize this region.

Step 5: Parameterization and Integration

Since the region is symmetric and bounded by a sphere and a plane, it’s convenient to use spherical coordinates:

$$
x = r \sin \phi \cos \theta, \quad y = r \sin \phi \sin \theta, \quad z = r \cos \phi
$$

with:

- $r \in [0,1]$ (since the sphere has radius 1),
- $\phi \in [0, \pi]$,
- $\theta \in [0, 2\pi]$.

The plane $x + y + z = 1$ becomes:

$$
r \sin \phi \cos \theta + r \sin \phi \sin \theta + r \cos \phi = 1
$$
$$
r (\sin \phi (\cos \theta + \sin \theta) + \cos \phi) = 1
$$

The boundary of the volume within the sphere is given by the set of points satisfying:

$$
r \left(\sin \phi (\cos \theta + \sin \theta) + \cos \phi \right) \leq 1
$$

To find the limits of integration explicitly, we need to analyze the intersection. However, this can be complex, so an alternative approach is to consider the symmetry and perhaps switch to a coordinate system aligned with the plane.

Step 6: Simplification Using Symmetry and Approximate Approach

Due to the symmetry of the problem, and assuming the sphere is centered at the origin with radius 1, the integration over y can be simplified by exploiting symmetry.

- The divergence y is an odd function with respect to y, and the volume is symmetric about the plane $y=0$.

- If the region V is symmetric with respect to $y=0$, then the volume integral of y over V is zero.

But, the plane $x + y + z = 1$ shifts the symmetry, so this may not be the case.

Step 7: Direct Calculation of the Volume Integral

Given the divergence y, the flux becomes:

$$
\Phi = \iiint_{V} y \, dV
$$

Because the divergence theorem simplifies the flux calculation, and the volume V is bounded by the sphere and the plane, we can:

- Use an appropriate coordinate system to parametrize V.
- Set up the triple integral accordingly.

Alternatively, if calculations become too cumbersome, the problem can be approached via the Surface Integral directly, parameterizing the surface S.

Step 8: Parameterizing the Surface S

Let's parameterize the surface S directly:

- The surface is part of the sphere $x^2 + y^2 + z^2 = 1$,
- bounded by the plane $x + y + z = 1$.

Solve for z:

$$
z = 1 - x - y
$$

Substitute into the sphere equation:

$$
x^2 + y^2 + (1 - x - y)^2 = 1
$$

Expand:

$$
x^2 + y^2 + (1 - x - y)^2 = 1
$$
$$
x^2 + y^2 + (1 - 2x - 2y + x^2 + 2xy + y^2) = 1
$$
$$
x^2 + y^2 + 1 - 2x - 2y + x^2 + 2xy + y
$$

4 Compute The Flux Of The Vector Fieldf X Y Z Yz Xz Yz Through The Part Of The Sphere X Y

4 Compute The Flux Of The Vector Fieldf X Y Z Yz Xz Yz Through The Part Of The Sphere X Y

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