4. Find The Solution To The Differential Equation Y"(t) + 5y'(t) + 2y(t) = 3u(t), Where Y(0) = A And

Understanding the Differential Equation: Y"(t) + 5y'(t) + 2y(t) = 3u(t), Where Y(0) = A

4. Find The Solution To The Differential Equation Y"(t) + 5y'(t) + 2y(t) = 3u(t), Where Y(0) = A And is a classic problem in differential equations, often encountered in engineering and physics. This type of second-order linear differential equation with constant coefficients models a variety of physical systems, such as mechanical vibrations, electrical circuits, and control systems.

In this article, we'll explore the steps to solve this differential equation comprehensively. Whether you're a student, educator, or professional, understanding the solution process will deepen your grasp of differential equations and their applications.

Overview of the Differential Equation Components

Understanding the Terms

- **Y"(t):** The second derivative of y(t), representing acceleration or curvature depending on the context.
- **Y'(t):** The first derivative of y(t), reflecting velocity or rate of change.
- **y(t):** The function we aim to find.
- **3u(t):** The forcing function, where u(t) is the unit step function (Heaviside function). This indicates a sudden change or input at t=0.
- **Y(0) = A:** The initial condition specifying the value of y(t) at t=0.

Implications of the Forcing Function

The presence of the unit step function u(t) makes the differential equation a nonhomogeneous equation. Its effect is to introduce a constant input starting at t=0, which influences the particular solution.

Step 1: Formulate the Homogeneous Equation

To find the general solution, begin by solving the homogeneous differential equation:

$$Y''(t) + 5Y'(t) + 2Y(t) = 0$$

This homogeneous equation reflects the system's natural response without external forcing.

Step 2: Solve the Characteristic Equation

Deriving the Characteristic Equation

Assuming solutions of the form $y(t) = e^{rt}$, substitute into the homogeneous differential equation:

$$r^2 e^{rt} + 5r e^{rt} + 2 e^{rt} = 0$$

Dividing through by e^{rt} (which is never zero), we get:

$$r^2 + 5r + 2 = 0$$

Solving for r

Apply the quadratic formula:

$$r = \frac{-5 \pm \sqrt{(5)^2 - 4 \times 1 \times 2}}{2}$$

$$r = \frac{-5 \pm \sqrt{25 - 8}}{2} = \frac{-5 \pm \sqrt{17}}{2}$$

Roots of the Characteristic Equation

- $r_1 = \frac{-5 + \sqrt{17}}{2}$
- $r_2 = \frac{-5 - \sqrt{17}}{2}$

Step 3: Write the General Homogeneous Solution

The general solution to the homogeneous equation is:

$$y_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

where C_1 and C_2 are constants determined by initial conditions.

Step 4: Find the Particular Solution

Addressing the Nonhomogeneous Term

The nonhomogeneous term is $3u(t)$, which is a constant input for $t \geq 0$. To find the particular solution, $y_p(t)$, we typically assume a form based on the forcing function.

Assumption for the Particular Solution

- If the forcing term is a constant, a good initial guess is $y_p(t) = K$, a constant.
- Alternatively, for differential equations, a constant particular solution is often effective when the right side is a constant.

Determine K by Substitution

Assuming $y_p(t) = K$, then:

$$Y'(t) = 0, \quad Y''(t) = 0$$

Substitute into the differential equation:

$$0 + 50 + 2K = 3u(t)$$

For $t \geq 0$, $u(t) = 1$, so:

$$2K = 3$$

$$K = \frac{3}{2}$$

Step 5: Write the General Solution

The overall general solution combines the homogeneous and particular solutions:

$$y(t) = y_h(t) + y_p(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{3}{2}$$

Step 6: Apply Initial Conditions to Determine Constants

Using Y(0) = A

At t=0, the solution becomes:

$$A = C_1 e^{r_1 \times 0} + C_2 e^{r_2 \times 0} + \frac{3}{2}$$

$$A = C_1 + C_2 + \frac{3}{2}$$

Find Y'(t) for initial velocity condition (if given)

In many problems, an initial velocity condition Y'(0) = B is provided. If so, differentiate the general solution:

$$Y'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t}$$

At t=0:

$$Y'(0) = C_1 r_1 + C_2 r_2$$

Using the initial velocity condition Y'(0) = B, solve for C_1 and C_2.

Summary of the Solution Process

1. Formulate the homogeneous differential equation and solve its characteristic equation.
2. Write the homogeneous solution based on the roots.
3. Identify the nonhomogeneous term and assume a particular solution accordingly.
4. Calculate the particular solution, often a constant in this case.
5. Combine the homogeneous and particular solutions to obtain the general solution.
6. Apply initial conditions to solve for constants.

Additional Considerations

Handling Discontinuities and Step Inputs

The presence of the unit step function u(t) indicates that the input begins at t=0. This often leads to a discontinuity or a sudden change in the system's response. To handle such cases, the Laplace Transform method is highly effective.

Laplace Transform Method

Applying the Laplace transform converts the differential equation into an algebraic equation, simplifying the process of solving for y(t). Here’s a brief outline:

- Take the Laplace transform of each term, considering initial conditions.
- Solve the algebraic equation for Y(s), the Laplace transform of y(t).
- Use inverse Laplace transform to find y(t).

Using Laplace Transform to Solve the Differential

Equation

Transform the Equation

```
L{Y"(t)} + 5 L{Y'(t)} + 2 L{Y(t)} = 3 L{u(t)}
```

Recall that:

- L{Y"(t)}

  # Frequently Asked Questions

  ## What is the general approach to solving the differential equation Y''(t) + 5Y'(t) + 2Y(t) = 3u(t)?

  The standard approach involves finding the complementary (homogeneous) solution by solving Y'' + 5Y' + 2Y = 0 and then determining a particular solution using methods such as undetermined coefficients or Laplace transforms, especially considering the unit step function u(t).

  ## How do initial conditions like Y(0) = A influence the solution of the differential equation?

  Initial conditions like Y(0) = A are used to determine the arbitrary constants in the homogeneous and particular solutions, ensuring the solution satisfies the given initial value problem.

  ## What is the role of the unit step function u(t) in this differential equation?

  The unit step function u(t) indicates the forcing term is active for t ≥ 0, making the problem a causal one and often leading to solutions involving Laplace transforms for simplicity.

  ## Can Laplace transforms be used to solve this differential

equation efficiently?

Yes, applying Laplace transforms simplifies the differential equation to algebraic form, especially when dealing with initial conditions and the unit step function, making it an efficient method for solving.

What is the form of the particular solution for the differential equation $Y'' + 5Y' + 2Y = 3u(t)$?

The particular solution can be found by taking the Laplace transform of both sides, solving for $Y(s)$, then taking the inverse Laplace transform. Since the right side involves $u(t)$, the particular solution often involves a constant term divided by the characteristic polynomial.

How does the initial condition $Y(0) = A$ affect the final solution after applying Laplace transforms?

The initial condition $Y(0) = A$ provides the initial value for $Y(t)$ at $t=0$, which is used to solve for constants in the inverse Laplace transform, ensuring the solution matches the initial state of the system.

Additional Resources

Comprehensive Analysis and Solution to the Differential Equation $Y''(t) + 5Y'(t) + 2Y(t) = 3u(t)$

Introduction

Differential equations serve as fundamental tools in modeling numerous physical, engineering, and scientific phenomena. They capture the relationship between functions and their derivatives, enabling us to analyze dynamic systems ranging from mechanical vibrations to electrical circuits. Among these, linear second-order differential equations with constant coefficients are particularly prominent due to their tractability and wide applicability.

In this detailed exploration, we focus on solving the specific differential equation:

$$
Y''(t) + 5Y'(t) + 2Y(t) = 3u(t),
$$

where $u(t)$ is the unit step function, and the initial condition $Y(0) = A$ is provided. This problem encapsulates several core concepts, including homogeneous and particular solutions, initial conditions, and the use of Laplace transforms, making it an excellent case study to understand the systematic approach to solving linear differential equations with nonhomogeneous terms.

Understanding the Differential Equation

The Equation Structure

- Type: Second-order linear differential equation with constant coefficients.
- Homogeneous part: $Y''(t) + 5Y'(t) + 2Y(t) = 0$.
- Nonhomogeneous part: The right side $3u(t)$ acts as a forcing function, representing an external input or disturbance, turned on at $t=0$.
- Initial condition: $Y(0) = A$, which influences the particular solution's constants.

Significance of the Unit Step Function $u(t)$

- The unit step function $u(t)$ equals 0 for $t < 0$ and 1 for $t \geq 0$.
- It models a sudden application of a force or input at $t=0$.
- The presence of $u(t)$ ensures the solution accounts for the system's response starting from $t=0$.

Step 1: Homogeneous Solution

The first step is to analyze the homogeneous differential equation:

$$
Y''(t) + 5Y'(t) + 2Y(t) = 0.
$$

Characteristic Equation:

$$
r^2 + 5r + 2 = 0.
$$

Finding Roots:

Applying the quadratic formula:

$$
r = \frac{-5 \pm \sqrt{(5)^2 - 4 \times 1 \times 2}}{2} = \frac{-5 \pm \sqrt{25 - 8}}{2} = \frac{-5 \pm \sqrt{17}}{2}.
$$

- Roots are real and distinct:

$$
r_1 = \frac{-5 + \sqrt{17}}{2}, \quad r_2 = \frac{-5 - \sqrt{17}}{2}.
$$

Homogeneous Solution:

$$
Y_h(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t},
$$

where C_1 and C_2 are constants determined by initial conditions.

Step 2: Particular Solution

Since the nonhomogeneous term is a constant $3u(t)$, the particular solution $Y_p(t)$ can be guessed as a constant, say $Y_p(t) = Y_p$, because the right side is constant.

Testing the constant guess:

$$
Y''(t) = 0, \quad Y'(t) = 0,
$$
so the differential equation reduces to:

$$
0 + 0 + 2 Y_p = 3,
$$

which yields:

$$
Y_p = \frac{3}{2}.
$$

Note: Since the forcing function is turned on at $t=0$, the particular solution applies for $t \geq 0$, and the system's steady-state response is $Y_p(t) = 3/2$.

Step 3: Applying Initial Conditions

The general solution for $t \geq 0$:

$$
Y(t) = Y_h(t) + Y_p = C_1 e^{r_1 t} + C_2 e^{r_2 t} + \frac{3}{2}.
$$

Given $Y(0) = A$, substitute $t=0$:

$$
A = C_1 + C_2 + \frac{3}{2}.
$$

To determine C_1 and C_2, we need the initial derivative $Y'(0)$. Let's find $Y'(t)$:

$$
Y'(t) = C_1 r_1 e^{r_1 t} + C_2 r_2 e^{r_2 t}.
$$

At $t=0$:

$$
Y'(0) = C_1 r_1 + C_2 r_2.
$$

However, since the initial derivative $Y'(0)$ is not specified, we might need to determine it using Laplace transforms, which naturally incorporate initial conditions.

Step 4: Solution via Laplace Transform

The Laplace transform method offers a systematic approach, especially with the step function involved.

Applying the Laplace Transform

- Recall that:

$$
\mathcal{L}\{Y(t)\} = Y(s),
$$
$$
\mathcal{L}\{Y'(t)\} = s Y(s) - Y(0),
$$
$$
\mathcal{L}\{Y''(t)\} = s^2 Y(s) - s Y(0) - Y'(0),
$$
- The transform of $u(t)$ is $\frac{1}{s}$.

Since $Y(0) = A$, and $Y'(0)$ remains unknown, the initial derivative can be considered as an additional unknown for now.

Transform the entire differential equation:

$$
s^2 Y(s) - sA - Y'(0) + 5 (s Y(s) - A) + 2 Y(s) = \frac{3}{s}.
$$

Rearranged:

$$
(s^2 + 5s + 2) Y(s) - sA - Y'(0) - 5A = \frac{3}{s}.
$$

Express $Y(s)$:

$$
Y(s) = \frac{\frac{3}{s} + sA + Y'(0) + 5A}{s^2 + 5s + 2}.
$$

This expression contains the unknown initial derivative $Y'(0)$. To proceed, assume the initial derivative is zero ($Y'(0) = 0$), which is common if not specified, or derive it based on physical considerations.

Step 5: Computing the Inverse Laplace Transform

Assuming $Y'(0) = 0$, the Laplace domain solution simplifies to:

$$
Y(s) = \frac{\frac{3}{s} + sA + 5A}{s^2 + 5s + 2}.
$$

Express numerator terms over a common denominator s:

$$
Y(s) = \frac{3 + (s^2 + 5A s)}{s (s^2 + 5s + 2)}.
$$

Break into partial fractions for easier inverse transformation.

Partial Fraction Decomposition:

$$
Y(s) = \frac{N(s)}{s D(s)},
$$

where:

- $N(s) = 3 + s^2 + 5A s$,
- $D(s) = s^2 + 5s + 2$.

Factor $D(s)$:

$$
s^2 + 5s + 2 = (s + r_1)(s + r_2),
$$

with roots:

$$
r_{1,2} = \frac{-5 \pm \sqrt{17}}{2}.
$$

Express $Y(s)$ as:

$$
Y(s) = \frac{A_1}{s} + \frac{A_2}{s + r_1} + \frac{A_3}{s + r_2}.
$$

Determine A_1, A_2, A_3 by standard partial fraction methods or residue calculation.

Step 6: Final Solution Expression

Once the partial fractions are identified, invert each term:

- The inverse Laplace of $\frac{1}{s}$ is $u(t)$,
- The inverse of $\frac{1}{s + r}$ is $e^{-r t}$.

Thus, the solution $Y(t)$ combines terms corresponding to the homogeneous solution and the forced response due to the step input.

Summary of the Solution Process

1. Homogeneous solution: Derived from characteristic roots.
2. Particular solution: Guessed as a constant, verified by substitution.
3. Initial conditions: Used to determine constants once the initial derivative is known or assumed.
4. Laplace transform method: Facilitates handling the step input and initial conditions systematically.
5. Partial fractions and inverse transform: Yield explicit time-domain solutions.

Physical Interpretation and Practical Considerations

- The homogeneous response reflects the system's natural behavior, typically dec

[4 Find The Solution To The Differential Equation Y T 5y T 2y T 3u T Where Y 0 A And]

4 Find The Solution To The Differential Equation Y T 5y T 2y T 3u T Where Y 0 A And

Related Articles

- A Cellphone Charger Has A Transformer With 2000 Turns In The Primary Coil Which Converts 240v A.c To
- A Concave Cosmetic Mirror Has A Focal Length Of 36 Cm . A 5.0-cm -long Mascara Brush Is Held Upright
- #1) You Are Tasked With Selecting The Supplier For A New Part From Four Possible Venders A,B, C, And

Back to Home