# 5. (a) (5 Points) Decide The Convergence Or Divergence Of The Sequence An = (-1)" And The Series An.

## 5. (a) (5 Points) Decide The Convergence Or Divergence Of The Sequence An = (-1)^n And The Series An

The problem at hand involves analyzing both a sequence and a series: specifically, the sequence $A_n = (-1)^n$ and the associated series $\sum_{n=1}^{\infty} A_n$. Determining whether this sequence converges or diverges, and whether the series converges or diverges, is fundamental in understanding the behavior of alternating sequences and their sums. This analysis requires applying core concepts from mathematical analysis, such as the criteria for convergence of sequences and series, the behavior of oscillating sequences, and the tests for convergence of infinite series. In this article, we will explore the properties of $A_n$ in detail and evaluate the convergence or divergence of both the sequence and the series.

## Understanding the Sequence $A_n = (-1)^n$

### Definition and Behavior of the Sequence

The sequence $A_n = (-1)^n$ is a classic example of an alternating sequence. It alternates between two values depending on whether $n$ is odd or even:

- For even $n$, $A_n = (-1)^{2k} = 1$ where $k \in \mathbb{N}$
- For odd $n$, $A_n = (-1)^{2k+1} = -1$

This means the sequence looks like:

$$A_1 = -1,\quad A_2 = 1,\quad A_3 = -1,\quad A_4 = 1,\quad \dots$$

### Limit of the Sequence $A_n$

To determine whether the sequence converges, we consider the limit of $A_n$ as $n \to \infty$:

$$ \lim_{n \to \infty} (-1)^n $$

Since the sequence oscillates between -1 and 1 without approaching a single value, the limit does not exist. Formally:

- As $n \to \infty$, the subsequence of even terms approaches 1.
- As $n \to \infty$, the subsequence of odd terms approaches -1.

Because the limits of these subsequences are not equal, the sequence $A_n$ does not have a limit. Therefore, we conclude:

### Conclusion: The sequence $A_n$ diverges.

## Analyzing the Series $\sum_{n=1}^\infty A_n = \sum_{n=1}^\infty (-1)^n$

### Definition of the Series

The series associated with the sequence is:

$$ \sum_{n=1}^\infty (-1)^n = -1 + 1 - 1 + 1 - 1 + \dots $$

This is an infinite alternating sequence of -1 and 1. The key question is whether the sum of this infinite series converges to a finite value or diverges.

### Test for Convergence of Series

To analyze the convergence of an infinite series, we commonly use the following tests:

1. **Test for Divergence (Nth-Term Test)**
2. **Alternating Series Test (Leibniz Criterion)**
3. **Comparison and Absolute Convergence Tests**

## Applying the Nth-Term Test for Divergence

The Nth-Term Test states that if $\lim_{n \to \infty} A_n \neq 0$, then the series $\sum A_n$ diverges.

Since $\lim_{n \to \infty} (-1)^n$ does not exist, but in particular, the terms do not tend to zero, the series cannot converge. In fact, because the terms oscillate between -1 and 1, their limit does not approach zero.

Therefore:

## Conclusion: The series diverges by the Nth-Term Test.

## Applying the Alternating Series Test

The Alternating Series Test requires that:

- The absolute value of the terms decreases monotonically to zero.
- $\lim_{n \to \infty} |A_n| = 0$

In our case:

- $|A_n| = 1$ for all $n$, which does not tend to zero.

Since the absolute value does not tend to zero, the series does not satisfy the conditions for convergence under the Alternating Series Test. Hence, the series diverges.

## Summary of Series Behavior

- The sequence $A_n = (-1)^n$ diverges because it oscillates and has no limit.
- The series $\sum_{n=1}^\infty (-1)^n$ diverges because its terms do not tend to zero, violating a fundamental necessary condition for convergence.

# Implications and Broader Context

### Why Does the Sequence Not Converge?

The core reason the sequence $A_n = (-1)^n$ does not converge is the lack of a single limit point. The sequence keeps oscillating between two fixed values, 1 and -1, indefinitely. Such behavior is characteristic of divergent sequences, especially those that do not settle into a particular value. This kind of oscillation indicates that the sequence does not stabilize and hence diverges.

### Why Does the Series Diverge?

In infinite series, a necessary condition for convergence is that the sequence of terms tends to zero. Since in our series the terms do not tend to zero, the sum cannot approach a finite limit. Consequently, the series diverges. Additionally, the partial sums oscillate between different values, preventing the series from having a finite sum.

## Additional Insights and Related Concepts

### Alternating Sequences and Series

Alternating sequences often appear in mathematical analysis, and their convergence properties are well-studied. The Alternating Series Test is a powerful tool for establishing convergence when the terms decrease monotonically to zero. However, in the case of $A_n = (-1)^n$, the terms do not tend to zero, so the test does not indicate convergence.

### Conditional and Absolute Convergence

- **Absolute Convergence:** A series $\sum a_n$ converges absolutely if $\sum |a_n|$ converges.
- **Conditional Convergence:** When a series converges but does not converge absolutely.

In our case, $\sum |A_n| = \sum 1$ diverges, so the series does not converge absolutely. Since the original series also diverges, it does not converge conditionally either.

## Summary and Final Remarks

To conclude, the analysis of the sequence $A_n = (-1)^n$ and the series $\sum_{n=1}^\infty (-1)^n$ reveals the following:

- The sequence \(A_n\) diverges because it oscillates between -1 and 1 and has no limit.
- The series \(\sum_{n=1}^\infty (-1)^n\) diverges because its terms do not tend to zero, violating a fundamental criterion for convergence.

This example illustrates the importance of the limit of the individual terms in assessing the convergence of series, and showcases common behavior in alternating sequences and series.

# Frequently Asked Questions

## What is the general form of the sequence An = (-1)^n, and how does its behavior affect convergence?

The sequence An = (-1)^n alternates between 1 and -1 indefinitely, so it does not approach a single value, indicating divergence.

## Does the sequence An = (-1)^n converge or diverge as n approaches infinity?

The sequence diverges because it oscillates between 1 and -1 and does not settle at a single limit.

## How can we determine the convergence of the series Σ An where An = (-1)^n?

Since An does not tend to zero (the necessary condition for series convergence), the series Σ (-1)^n diverges.

## Is the series Σ (-1)^n convergent? Why or why not?

No, the series diverges because its terms do not approach zero; the partial sums oscillate between different values.

## What is the behavior of the partial sums of the series Σ (-1)^n?

The partial sums oscillate between 0 and 1, indicating that the series does not converge.

## Can the series Σ (-1)^n be considered an alternating series? Why or why

## not?

Yes, it is an alternating series because its terms alternate in sign, but it still diverges due to the terms not approaching zero.

## What is the key criterion for the convergence of a series, and does $\Sigma (-1)^n$ satisfy it?

The key criterion is that the terms must approach zero; since $(-1)^n$ does not, the series diverges.

# Additional Resources

Deciding the Convergence or Divergence of the Sequence $A_n = (-1)^n$ and the Series $\sum A_n$

Understanding the behavior of sequences and series is fundamental in mathematical analysis, especially in the study of limits, convergence, and divergence. The sequence $A_n = (-1)^n$ is a classic example frequently used to illustrate the concepts of oscillation and divergence. Analyzing this sequence and its corresponding series provides insights into the criteria that determine convergence or divergence.

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# Overview of Sequences and Series

Before delving into the specifics of $A_n = (-1)^n$, it is essential to clarify the definitions and distinctions between sequences and series:

- Sequence: An ordered list of terms $\{A_n\}$ indexed by natural numbers $n$. The focus is on the behavior of individual terms as $n$ approaches infinity.
- Series: An infinite sum of the form $\sum_{n=1}^\infty a_n$, where $a_n$ is a sequence. The primary question is whether the sum converges to a finite limit or diverges to infinity or oscillates indefinitely.

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# Analyzing the Sequence $A_n = (-1)^n$

## Definition and Initial Observations

The sequence in question:

$$
A_n = (-1)^n
$$

- For even $n$: $A_n = (-1)^{2k} = 1$
- For odd $n$: $A_n = (-1)^{2k+1} = -1$

Thus, the sequence alternates between 1 and -1:

$$
A_1 = -1, \quad A_2 = 1, \quad A_3 = -1, \quad A_4 = 1, \quad \ldots
$$

Initial observations:

- The sequence does not approach a single value as $n \to \infty$.
- The terms oscillate indefinitely between -1 and 1.
- The limit $\lim_{n \to \infty} A_n$ does not exist.

## Limit of the Sequence $A_n$

To determine whether the sequence converges:

$$
\lim_{n \to \infty} (-1)^n
$$

Since the sequence oscillates between 1 and -1, the limit does not exist. More formally:

- The sequence does not settle to a single value.
- The subsequences:

$$
A_{2k} = 1 \quad \text{and} \quad A_{2k+1} = -1
$$

do not approach the same limit.

Conclusion: The sequence $A_n = (-1)^n$ diverges.

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# Convergence of the Series $\sum_{n=1}^\infty A_n = \sum_{n=1}^\infty (-1)^n$

While the behavior of the sequence itself indicates divergence, the series formed by summing its terms warrants another analysis.

## Definition of Series and Convergence Criteria

- The series:

$$
S = \sum_{n=1}^\infty a_n = a_1 + a_2 + a_3 + \ldots
$$

- The series converges if the sequence of partial sums:

$$
S_N = \sum_{n=1}^N a_n
$$

approaches a finite limit as $N \to \infty$. Otherwise, it diverges.

## Partial Sums of the Series $\sum (-1)^n$

Let's examine the partial sums:

$$
S_N = \sum_{n=1}^N (-1)^n
$$

Calculate the first few partial sums:

- $N=1$:

$$
S_1 = (-1)^1 = -1
$$

- $N=2$:

$$
S_2 = (-1) + 1 = 0
$$

- $N=3$:

$$
S_3 = -1 + 1 - 1 = -1
$$

- $N=4$:

$$
S_4 = -1 + 1 - 1 + 1 = 0
$$

Similarly, for larger $N$:

- For odd $N$:

$$
S_N = -1
$$

- For even $N$:

$$
S_N = 0
$$

Observation:

- The sequence of partial sums oscillates between -1 and 0.
- The partial sums do not approach a single finite value but oscillate indefinitely.

## Does the Series Converge?

Since the partial sums oscillate between -1 and 0, they do not tend to a finite limit. Therefore:

- The series $\sum_{n=1}^\infty (-1)^n$ diverges.

## Summary of Series Behavior

| Property | Result |
| --- | --- |
| Sequence $\{A_n\}$ | Diverges (oscillates between -1 and 1) |
| Partial sums $\{S_N\}$ | Oscillate between -1 and 0 |
| Series $\sum_{n=1}^\infty (-1)^n$ | Diverges |

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# Deeper Insights: Tests for Convergence and Divergence

## Limit Test

- Sequence: The limit $\lim_{n \to \infty} A_n$ does not exist.
- Series: For a series to converge, the general term must tend to zero (a necessary condition). Here:

$$
\lim_{n \to \infty} a_n = \lim_{n \to \infty} (-1)^n \text{ does not exist.}
$$

- Conclusion: The series cannot converge, as the terms do not tend to zero.

## Term Test for Divergence

- The Term Test states:

> If $\lim_{n \to \infty} a_n \neq 0$, then $\sum a_n$ diverges.

- Here:

$$
\lim_{n \to \infty} (-1)^n \text{ does not exist, hence not zero.}
$$

- Therefore, the series:

$$
\sum_{n=1}^\infty (-1)^n
$$

diverges.

## Alternating Series Test (Leibniz Test)

This test states:

> An alternating series $\sum (-1)^n b_n$ converges if:

> 1. $b_n$ is decreasing: $b_{n+1} \leq b_n$
> 2. $\lim_{n \to \infty} b_n = 0$

In our case:

$$
a_n = (-1)^n \implies b_n = 1
$$

- $b_n$ is constant at 1, which is not decreasing.
- $\lim_{n \to \infty} b_n = 1 \neq 0$.

Thus, the alternating series test fails.

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# Summary and Key Takeaways

- The sequence $A_n = (-1)^n$ diverges because it oscillates between 1 and -1 and does not approach a limit.
- The series $\sum_{n=1}^\infty (-1)^n$ diverges because its partial sums oscillate and do not approach a finite value.

- The necessary condition for series convergence—that the general term tends to zero—is not satisfied here.
- The sequence's divergence is a clear indicator that the series cannot converge.

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## Implications and Broader Context

The sequence $A_n = (-1)^n$ is a canonical example demonstrating the importance of understanding the behavior of both sequences and series. It helps illustrate:

- The critical role of the limit of the terms in determining convergence.
- The fact that oscillation alone does not necessarily imply convergence of the sequence or series.
- The necessity of the limit of the sequence's terms being zero for the convergence of the associated series.
- Why oscillating sequences often lead to divergence, especially when their limits do not exist or are non-zero.

This example also underscores the importance of various convergence tests in real analysis, such as the Limit Test, the Term Test, and the Alternating Series Test, which provide systematic ways to analyze and determine convergence properties.

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## Conclusion

In conclusion, the sequence $A_n = (-1)^n$ diverges because it oscillates indefinitely and lacks a finite limit. Correspondingly, the series $\sum_{n=1}^\infty (-1)^n$ also diverges due to the oscillating partial sums and the failure to satisfy the basic convergence criteria. This example serves as a classic illustration in analysis of how oscillatory behavior influences the convergence or divergence of sequences and series, emphasizing the importance of

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