5. Consider An LTI System With Input X[n] And Output Y[n] For Which Y[n]y[n 1] +y[n - 2] = X[n]. The

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Introduction

In the field of signal processing, Linear Time-Invariant (LTI) systems are fundamental because of their simplicity and powerful analysis tools. These systems are characterized by their linearity and invariance with respect to shifts in time, which means their behavior does not change over time and the principle of superposition applies. Understanding the behavior of LTI systems, especially those described by difference equations, is crucial for designing filters, analyzing signals, and understanding system responses.

This article explores an LTI system described by the difference equation:

$$ Y[n] y[n-1] + y[n-2] = X[n] $$

where $X[n]$ is the input signal and $Y[n]$ is the output signal. We will analyze this system in detail, exploring its properties, solution methods, impulse response, frequency response, and stability considerations.

Understanding the System Equation

The Difference Equation

The given difference equation:

$$ y[n] y[n-1] + y[n-2] = x[n] $$

is a nonlinear form because of the term $y[n] y[n-1]$. However, in typical LTI system analysis, the difference equations are linear, involving sums and scalar multiples of shifted signals.

Note: If the original intended equation was:

$$ y[n] = y[n-1] + y[n-2] + x[n] $$

or similar linear forms, the analysis simplifies significantly.

Assuming the original equation is:

$$ y[n] y[n-1] + y[n-2] = x[n] $$

which is nonlinear, but for the purpose of this discussion, we'll interpret the equation as a linear difference equation of the form:

$$ y[n] = -a_1 y[n-1] - a_2 y[n-2] + x[n] $$

and proceed accordingly.

Alternatively, if the equation is linear but written differently, the analysis proceeds as follows.

Analyzing a Typical LTI Difference Equation

Assuming the difference equation is:

$$ y[n] + a_1 y[n-1] + a_2 y[n-2] = x[n] $$

where a_1 and a_2 are constants, and the system is causal.

This form represents a second-order linear difference equation, which is common in digital filter design.

Solving the Homogeneous Equation

The first step in analyzing such systems is to find the homogeneous solution:

$$ y_h[n] \text{ from } y[n] + a_1 y[n-1] + a_2 y[n-2] = 0 $$

The characteristic equation is:

$$ r^2 + a_1 r + a_2 = 0 $$

which has roots:

$$ r_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4 a_2}}{2} $$

Depending on the roots, the homogeneous solution varies:

- Distinct real roots:
$$ y_h[n] = C_1 r_1^{n} + C_2 r_2^{n} $$

- Repeated roots:
$$ y_h[n] = (D_1 + D_2 n) r^{n} $$

- Complex roots:
$$ y_h[n] = \alpha r^{n} \cos(\omega n) + \beta r^{n} \sin(\omega n) $$

where $C_1, C_2, D_1, D_2, \alpha, \beta$ are constants determined by initial conditions.

Finding the Particular Solution

To find the particular solution $y_p[n]$ for the nonhomogeneous equation, methods such as the Undetermined Coefficients or Z-transform are used.

- Undetermined Coefficients:
Suitable for inputs $x[n]$ such as constants, exponentials, or sinusoids.

- Z-Transform:
Converts difference equations into algebraic equations, making it easier to solve for $Y(z)$ and hence $y[n]$.

Using the Z-Transform

The Z-transform of the difference equation:

$$ y[n] + a_1 y[n-1] + a_2 y[n-2] = x[n] $$

becomes:

$$ Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) = X(z) $$

which simplifies to:

$$ Y(z) (1 + a_1 z^{-1} + a_2 z^{-2}) = X(z) $$

or

$$ Y(z) = \frac{X(z)}{1 + a_1 z^{-1} + a_2 z^{-2}} $$

The transfer function $H(z)$ is:

$$ H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2}} $$

Rearranged as:

$$ H(z) = \frac{z^2}{z^2 + a_1 z + a_2} $$

This form allows for analysis of the system's frequency response, poles, zeros, and stability.

Impulse Response and System Behavior

The impulse response $h[n]$ of the system is the inverse Z-transform of $H(z)$. It characterizes how the system responds to a unit impulse $\delta[n]$.

Calculating $h[n]$:

- Factor the denominator polynomial to find poles.
- Perform partial fraction expansion if necessary.
- Inverse Z-transform to obtain $h[n]$.

The impulse response provides insights into the system's stability and causality.

Frequency Response and Stability

The frequency response $H(e^{j\omega})$ is obtained by evaluating $H(z)$ on the unit circle $z = e^{j \omega}$:

$$H(e^{j \omega}) = \frac{e^{j 2 \omega}}{e^{j 2 \omega} + a_1 e^{j \omega} + a_2}$$

Stability Criterion:

- For BIBO (Bounded Input, Bounded Output) stability, all poles must lie inside the unit circle in the z-plane.
- The locations of roots of the characteristic polynomial determine the system's stability.

Practical Examples and Applications

Digital Filter Design

Understanding the difference equations and their transfer functions is crucial in designing digital filters like low-pass, high-pass, and band-pass filters.

Signal Analysis

Impulse and step responses reveal how signals propagate through systems and how signals are attenuated or amplified.

Control Systems

Difference equations model discrete-time control systems, and their stability analysis helps in controller design.

Summary and Key Takeaways

- Difference equations form the backbone of discrete-time LTI system analysis.
- Homogeneous solutions depend on the roots of the characteristic equation.
- Particular solutions depend on the nature of the input signal.

- The Z-transform simplifies solving difference equations and analyzing system behavior.
- The transfer function provides insights into the system's frequency response, stability, and causality.
- Impulse response characterizes the system's fundamental behavior.

Final Remarks

The analysis of LTI systems described by difference equations is essential for engineers and signal processing practitioners. Whether designing filters or analyzing signals, understanding the underlying mathematics enables precise control and prediction of system behavior. Although the initial equation provided appears nonlinear, the principles outlined here apply broadly to linear systems, which are the most common in practical applications.

References

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End of Article

Frequently Asked Questions

What is the difference equation of the given LTI system?

The difference equation is Y[n] y[n - 1] + y[n - 2] = X[n].

How can we find the system's transfer function H(z) for the given difference equation?

Taking Z-transform on both sides yields Y(z) [1 - z^{-1} - z^{-2}] = X(z), so H(z) = Y(z)/X(z) = 1 / (1 - z^{-1} - z^{-2}).

Is the system causal and stable based on the difference equation?

Yes, since the difference equation relates current output to past outputs and current input, it is causal. Stability depends on the roots of the characteristic equation; if all roots are inside the unit circle, the system is stable.

What is the characteristic equation associated with the homogeneous part of the system?

The characteristic equation is $y[n] - y[n - 1] - y[n - 2] = 0$, or equivalently, $r^2 - r - 1 = 0$.

How do the roots of the characteristic equation influence the system's behavior?

The roots determine the system's response; if roots are within the unit circle, the response is bounded and stable; if outside, the system is unstable.

Can you determine the impulse response y[n] of the system?

Yes, by solving the homogeneous equation with initial conditions and the particular solution corresponding to the input, the impulse response can be derived using inverse Z-transform methods.

What is the significance of the system being linear and time-invariant in this context?

Linearity and time-invariance ensure that the system's response to any input can be characterized using superposition and convolution, simplifying analysis and design.

How does the presence of feedback in the difference equation affect system stability?

The feedback terms (past outputs) influence the characteristic roots; if feedback leads roots outside the unit circle, the system becomes unstable. Proper analysis of roots is essential to assess stability.

Additional Resources

LTI System Analysis and Characterization: A Deep Dive into the Difference Equation Y[n]y[n-1] + y[n-2] = X[n]

Introduction

Linear Time-Invariant (LTI) systems are foundational in signal processing, control systems, and communications. Their properties, analysis techniques, and applications are essential knowledge for engineers and researchers. Today, we explore a specific discrete-time LTI system characterized by the difference equation:

\[
Y[n] y[n-1] + y[n-2] = X[n]
\]

This equation presents a fascinating case—combining linearity with a nonlinear-looking term \(Y[n]

$y[n-1]$. The way this equation is structured suggests some ambiguity, which we will clarify. Our goal is to analyze this system in depth, understand its behavior, derive its properties, and explore its response to various inputs.

Clarifying the System Equation

Before delving into the analysis, it's crucial to interpret the given difference equation properly. The original statement is:

$$
Y[n] y[n-1] + y[n-2] = X[n]
$$

At first glance, this seems to involve the output $Y[n]$ (probably intended as $y[n]$) and the input $X[n]$. However, the notation suggests that $Y[n]$ is the output, and the equation involves $Y[n]$ itself, which implies a nonlinear term because of the product $Y[n] y[n-1]$.

Likely correction:

- The notation may have a typo, and the intended equation is:

$$
y[n] y[n-1] + y[n-2] = x[n]
$$

or

- The equation is linear in $y[n]$, possibly:

$$
a_1 y[n-1] + a_2 y[n-2] = x[n]
$$

or

- The notation is consistent with a typical difference equation:

$$
Y[n] y[n-1] + y[n-2] = X[n]
$$

but more plausibly, the correct form intended is:

$$
a_1 y[n-1] + a_2 y[n-2] = x[n]
$$

which is a standard second-order linear difference equation.

Assumption: The System is Linear and Time-Invariant

Given the context, and since the prompt emphasizes an LTI system, we will assume that the intended difference equation is:

$$
a_1 y[n-1] + a_2 y[n-2] = x[n]
$$

where a_1, a_2 are constants.

Suppose, for the sake of illustration, the specific difference equation is:

$$
y[n-1] + 2 y[n-2] = x[n]
$$

which is a common form encountered in system analysis.

System Characterization

1. System Difference Equation

The core of the analysis is the difference equation:

$$
a_1 y[n-1] + a_2 y[n-2] = x[n]
$$

This recursive relation defines how the output $y[n]$ depends on past outputs and the current input. It is linear because the output appears linearly, and time-invariant because the coefficients a_1 and a_2 are constant.

---

2. Homogeneous Solution

The homogeneous part corresponds to solving:

$$
a_1 y[n-1] + a_2 y[n-2] = 0
$$

The characteristic equation is:

$$
a_1 r + a_2 = 0
$$

\]

which yields the roots:

$$
r = \frac{0}{a_1} \quad \text{(if } a_1 \neq 0\text{)} \quad \text{or} \quad \text{roots depending on the specific coefficients}
$$

For a second-order system, the characteristic equation becomes:

$$
a_2 r^2 + a_1 r = 0
$$

or equivalently:

$$
r (a_2 r + a_1) = 0
$$

with roots:

$$
r_1 = 0, \quad r_2 = - \frac{a_1}{a_2}
$$

The homogeneous solution is:

$$
y_h[n] = C_1 r_1^{n} + C_2 r_2^{n}
$$

where C_1, C_2 are constants determined by initial conditions.

---

3. Particular Solution

The particular solution depends on the input $x[n]$.

- For constant input $x[n] = X_0$, the steady-state response y_p is a constant:

$$
y_p = \frac{X_0}{a_1 + a_2}
$$

- For sinusoidal input $x[n] = A \cos(\omega n)$, the particular solution involves complex exponentials, leading to the system's frequency response.

---

4. System Response and Stability

The roots of the characteristic equation directly influence the system's stability:

- Stable if all roots have magnitude less than 1.
- Unstable if any root exceeds magnitude 1.
- Marginally stable if roots lie on the unit circle.

Understanding the roots helps predict whether the system's output will settle, oscillate, or diverge for a given input.

---

5. Impulse and Step Responses

- Impulse response $h[n]$: Derived by setting $x[n] = \delta[n]$, where $\delta[n]$ is the Kronecker delta.

- Step response: Obtained by setting $x[n] = u[n]$, the unit step.

These responses provide insight into the system's dynamics, such as transient behavior and steady-state output.

Practical Example: Solving a Specific System

Suppose the difference equation is:

$$
y[n-1] + 2 y[n-2] = x[n]
$$

with initial conditions $y[-1] = 0$, $y[-2] = 0$, and input $x[n] = \delta[n]$.

Step 1: Homogeneous solution

Characteristic equation:

$$
r^2 + 2 r = 0 \Rightarrow r(r + 2) = 0
$$

Roots:

$$
r_1 = 0, \quad r_2 = -2
$$

Homogeneous solution:

$$
y_h[n] = C_1 (0)^n + C_2 (-2)^n
$$

Since $0^n = 0$ for $n > 0$, the homogeneous solution simplifies to:

$$
y_h[n] = C_2 (-2)^n
$$

Step 2: Particular solution

For $x[n] = \delta[n]$, the particular solution $y_p[n]$ is the impulse response $h[n]$.

Applying the Z-transform:

$$
Y(z) = \frac{X(z)}{a_1 z^{-1} + a_2 z^{-2}}
$$

or more straightforwardly, solving recursively.

Step 3: Compute impulse response

At $n=0$:

$$
y[-1] = 0, \quad y[-2]=0, \quad x[0]=1
$$

Equation at $n=0$:

$$
y[-1] + 2 y[-2] = x[0] \Rightarrow 0 + 0=1
$$

which is inconsistent unless initial conditions are adjusted or the recursive calculation is performed carefully.

Alternatively, directly iterating:

- For $n=0$:

$$
y[0] \text{ depends on initial conditions and input}
$$

This illustrates the importance of initial conditions in the impulse response.

Frequency Response and System Behavior

The transfer function $H(z)$:

$$
H(z) = \frac{Y(z)}{X(z)} = \frac{1}{a_1 z^{-1} + a_2 z^{-2}}
$$

which, after clearing denominators, becomes:

$$
H(z) = \frac{z^2}{a_1 z + a_2}
$$

The frequency response $H(e^{j\omega})$ is obtained by substituting $z = e^{j\omega}$:

$$
H(e^{j\omega}) = \frac{e^{2 j \omega}}{a_1 e^{j \omega} + a_2}
$$

Analysis of magnitude and phase across the frequency spectrum reveals how the system filters signals, whether it amplifies certain frequencies or attenuates others.

Real-World Applications

Such second-order difference equations model numerous systems:

- Mechanical systems: mass-spring-damper models
- Electrical circuits: RLC filters
- Control systems: discrete controllers
- Economical models: discrete-time economic indicators

Understanding their solutions enables engineers to design filters, controllers, and systems with desired transient and steady-state behaviors.

Final Remarks: System Design and Stability

Designing an LTI system involves selecting coefficients a_1, a_2 to achieve specific properties:

- Stability: roots within the unit circle
- Response speed: controlled by pole locations
- Frequency selectivity:

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