5. Find Power Series Solution For The ODE About X = 0 In The Form Of Y=x_nx" =0 (x 4)y" + 3xy' + Y =

Introduction

The process of finding power series solutions to differential equations is a fundamental technique in mathematical analysis, especially when solutions cannot be expressed in closed form using elementary functions. This method is particularly useful for solving linear ordinary differential equations (ODEs) around ordinary points, where the coefficients are analytic functions. The given differential equation, which appears somewhat complex, can be approached effectively by assuming a power series solution centered at x=0. This approach involves expressing the unknown function as an infinite sum of powers of x and determining the coefficients systematically.

Understanding the Differential Equation

Given Equation

The differential equation provided is somewhat ambiguously formatted, but it appears to be:

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(x^4) y'' + 3x y' + y = 0
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This is a second-order linear homogeneous differential equation with variable coefficients. The coefficients involve powers of x, indicating that a power series solution around x=0 is appropriate, especially if x=0 is an ordinary point (i.e., the coefficients are analytic at x=0).

Preliminary Analysis

Identifying the Type of Equation

- The equation involves variable coefficients with polynomial expressions.
- The point x=0 is likely an ordinary point because the coefficients are polynomials and are analytic at x=0.
- The goal is to find a power series solution of the form:

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  Y(x) = Σ a_n x^n, n=0 to ∞
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Assumptions for the Power Series Solution

Assume:

$$Y(x) = \Sigma_{n=0}^\infty a_n x^n$$

where a_n are constants to be determined. The derivatives are:

$$Y'(x) = \Sigma_{n=1}^\infty n a_n x^{n-1}$$

$$Y''(x) = \Sigma_{n=2}^\infty n(n-1) a_n x^{n-2}$$

Deriving the Series Substitutions

Expressing y'', y', and y in Series Form

- For *y*'':

  $$y'' = \Sigma_{n=2}^\infty n(n-1) a_n x^{n-2}$$

- For *y*':

  $$y' = \Sigma_{n=1}^\infty n a_n x^{n-1}$$

- For *y*:

  $$y = \Sigma_{n=0}^\infty a_n x^n$$

Substituting these into the differential equation:

$$(x^4) y'' + 3x y' + y = 0$$

Substituting Series into the Differential Equation

Term-by-Term Substitution

1. **Term 1: $(x^4) y''$**

   $$x^4 y'' = x^4 \Sigma_{n=2}^\infty n(n-1) a_n x^{n-2} = \Sigma_{n=2}^\infty n(n-1) a_n x^{n+2}$$

2. **Term 2: $3x y'$**

   $$3x y' = 3x \Sigma_{n=1}^\infty n a_n x^{n-1} = 3 \Sigma_{n=1}^\infty n a_n x^n$$

3. **Term 3: y**

   $$y = \Sigma_{n=0}^\infty a_n x^n$$

Reindexing the Series for Consistency

- For the first term, the index is n starting at 2, but the powers of x are n+2. To write all series in terms of x^n, redefine the index:

$$\text{Let } m = n + 2 \Rightarrow n = m - 2$$

- So:

$$x^4 y'' = \Sigma_{m=4}^\infty (m-2)(m-3) a_{m-2} x^m$$

- For the second term, the series is already in powers of x^n starting at n=1.
- For the third term, series is in x^n, starting at n=0.

Formulating the Power Series Equation

Combining All Terms

$$\Sigma_{m=4}^\infty (m-2)(m-3) a_{m-2} x^m + 3 \Sigma_{n=1}^\infty n a_n x^n + \Sigma_{n=0}^\infty a_n x^n = 0$$

Aligning the Series

- To combine, write all sums in terms of a common index n:

For the first sum:
$m \to n$, starting from $n=4$:
$$\Sigma_{n=4}^{\infty} (n-2)(n-3)\, a_{n-2}\, x^n$$
For the second sum:
$n \to n$, from $n=1$:
$$3\, \Sigma_{n=1}^{\infty} n\, a_n\, x^n$$
For the third sum:
$n \to n$, from $n=0$:
$$\Sigma_{n=0}^{\infty} a_n\, x^n$$

Expressing the Entire Equation in a Unified Series

$$\Sigma_{n=4}^{\infty} (n-2)(n-3)\, a_{n-2}\, x^n + 3\, \Sigma_{n=1}^{\infty} n\, a_n\, x^n + \Sigma_{n=0}^{\infty} a_n\, x^n = 0$$

To combine these into a single summation, note the different starting points:
- For $n=0$ and $n=1$, only some sums contribute.
- For $n \geq 4$, all sums contribute.

Deriving the Recurrence Relations

Grouping Terms by Powers of x

- Rewrite the entire sum as:

$$a_0 + a_1 x + \Sigma_{n=2}^{\infty} [\, (n-2)(n-3)\, a_{n-2} + 3\, n\, a_n + a_n \,]\, x^n = 0$$

- For $n=0$:

$$a_0 = 0 \text{ (since the sum equals zero)}$$

- For $n=1$:

$$a_1 + 0 + 0 = 0 \Rightarrow a_1 = 0$$

- For $n \geq 2$:

$$(n-2)(n-3)\, a_{n-2} + (3\, n + 1)\, a_n = 0$$

Recurrence Formula

- Re-arranged:

a_n = - [(n-2)(n-3)] / (3 n + 1) a_{n-2}

- This recurrence allows calculating all coefficients based on initial values a_0 and a_1, which have been found to be zero, indicating the trivial solution unless further initial conditions are specified.

Solving for Coefficients

Initial Conditions

- Typically, for a power series solution, initial values a_0 and a_1 are chosen freely, reflecting the general solution.
- From the previous step:

a_0 = arbitrary, a_1 = arbitrary

- But based on the recurrence:

a_0 and a_1 are independent parameters, they can be assigned any value, leading to a family of solutions.

Calculating Subsequent Coefficients

- For n=2:

a_2 = - [(0)(-1)] / (7) a_0 = 0

- For n=3:

a_3 = - [(1)(0)] / (10) a_1 = 0

- For n=4:

a_4 = - [(2)(-1)] / (13) a_2 = 0 (since a_2=0)

- Similarly, all higher coefficients will depend on initial parameters, but in this case, the recurrence indicates that if a_0 and a_1 are zero, the entire solution is trivial unless other initial conditions are set.

Constructing the General Solution

Series Expression