A Closed Cylindrical Can Has A Fixed Surface Area S. Find The Ratio Of Its Height To The Diameter Of

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Introduction

A closed cylindrical can is a common packaging container used for storing liquids, foods, and other materials. Its design is optimized for strength, volume, and material efficiency. When manufacturing such cans, one key aspect to consider is the surface area, which directly influences the material cost and weight. Suppose the total surface area of the can is fixed at a value S; understanding the relationship between its height and diameter becomes essential for optimizing design and material usage.

In this article, we explore how to determine the ratio of the height to the diameter of a closed cylindrical can with a fixed surface area. We will delve into the mathematical formulation, derive the necessary equations, analyze the relationship, and understand the implications for practical design.

Understanding the Geometry of a Closed Cylindrical Can

Basic Dimensions and Notation

Consider a cylindrical can with:

- Radius: r
- Height: h
- Diameter: $d = 2r$

The key parameters are interrelated, and the goal is to express the ratio $\frac{h}{d}$, which simplifies to $\frac{h}{2r}$.

Surface Area of a Closed Cylinder

The total surface area S of a closed cylinder comprises:

- The lateral surface area: $2\pi r h$
- The area of the two circular ends: $2 \times \pi r^2$

Therefore, the total surface area is:

$$
S = 2\pi r h + 2 \pi r^2
$$

Given that S is fixed, the relationship between r and h can be explored.

Mathematical Formulation

Expressing the Surface Area Equation

Starting with:

$$
S = 2\pi r h + 2 \pi r^2
$$

Divide both sides by $2\pi r$:

$$
\frac{S}{2\pi r} = h + r
$$

Rearranged:

$$
h = \frac{S}{2\pi r} - r
$$

This expression links the height h and the radius r for a given fixed surface area S.

Objective: Find $\frac{h}{d}$

Recall that $d = 2r$, so:

$$
\frac{h}{d} = \frac{h}{2r}
$$

Substitute the expression for h:

$$
\frac{h}{d} = \frac{\frac{S}{2\pi r} - r}{2r}
$$

Simplify numerator and denominator:

$$
\frac{h}{d} = \frac{\frac{S}{2\pi r} - r}{2r} = \frac{\frac{S}{2\pi r}}{2r} - \frac{r}{2r} = \frac{S}{4\pi r^2} - \frac{1}{2}
$$

\]

Thus, the ratio \(\frac{h}{d} \) depends on the radius \(r \):

\[
\boxed{
\frac{h}{d} = \frac{S}{4 \pi r^2} - \frac{1}{2}
}
\]

Optimizing the Dimensions for a Given Surface Area

Goal

To understand the typical ratio of height to diameter, we can analyze the behavior of \(\frac{h}{d} \) with respect to \(r \).

Critical Point: Minimizing or Maximizing the Ratio

In practical scenarios, designers might want to optimize the height-to-diameter ratio for specific purposes, such as maximizing volume or minimizing material. To find the optimal \(r \), we can differentiate \(\frac{h}{d} \) with respect to \(r \) and analyze critical points.

Differentiation

Given:

\[
\frac{h}{d} = \frac{S}{4 \pi r^2} - \frac{1}{2}
\]

Differentiate with respect to \(r \):

\[
\frac{d}{dr} \left(\frac{h}{d} \right) = - \frac{S}{2 \pi r^3}
\]

Since \(S \), \(\pi \), and \(r \) are positive, the derivative is always negative, indicating that \(\frac{h}{d} \) decreases monotonically as \(r \) increases.

Implication

- As \(r \to 0 \), \(\frac{h}{d} \to +\infty \)
- As \(r \to \infty \), \(\frac{h}{d} \to -\frac{1}{2} \)

But since the can must have a positive height, the physically meaningful \(r \) must satisfy:

\[
h = \frac{S}{2\pi r} - r > 0

\]

which leads to:

\[
\frac{S}{2\pi r} > r \implies S > 2 \pi r^2
\]

This inequality constrains the feasible radius.

Practical Interpretation and Design Considerations

How Does the Ratio Change with Radius?

The ratio:

\[
\frac{h}{d} = \frac{S}{4 \pi r^2} - \frac{1}{2}
\]

shows that for a fixed surface area:

- Smaller radius \(r \) results in a larger \(\frac{h}{d} \) ratio, meaning a taller, narrower can.
- Larger radius \(r \) results in a smaller \(\frac{h}{d} \) ratio, leading to a shorter, wider can.

Optimal Design Strategies

Depending on the application, manufacturers might prefer:

- Tall, narrow cans for ease of stacking or aesthetic purposes.
- Short, wide cans for stability or volume considerations.

By choosing the radius \(r \) within the feasible range, the desired height-to-diameter ratio can be achieved.

Example Calculation

Suppose the total surface area \(S \) is known, for example:

\[
S = 1000\, \text{cm}^2
\]

Compute \(\frac{h}{d} \) for different values of \(r \):

- For \(r = 5\, \text{cm} \):

\[
\frac{h}{d} = \frac{1000}{4 \pi (5)^2} - \frac{1}{2} = \frac{1000}{4 \pi \times 25} - 0.5 \approx

$$
\frac{1000}{314.16} - 0.5 \approx 3.18 - 0.5 = 2.68
$$

- For $r = 10\, \text{cm}$:

$$
\frac{h}{d} \approx \frac{1000}{4 \pi \times 100} - 0.5 \approx \frac{1000}{1256.64} - 0.5 \approx 0.796 - 0.5 = 0.296
$$

These examples illustrate how increasing radius decreases the height-to-diameter ratio.

Summary and Conclusions

Key Takeaways

- The ratio of height to diameter of a closed cylindrical can with a fixed surface area S is given by:

$$
\boxed{
\frac{h}{d} = \frac{S}{4 \pi r^2} - \frac{1}{2}
}
$$

- This ratio depends inversely on the square of the radius r: increasing r decreases $\frac{h}{d}$.
- The feasible radius must satisfy $S > 2 \pi r^2$ to ensure a positive height.

Practical Implications

Understanding this relationship enables manufacturers and designers to:

- Optimize can dimensions based on material constraints.
- Tailor the shape for specific functional or aesthetic requirements.
- Balance between material cost (surface area) and internal volume.

Final Thoughts

Designing a cylindrical can with a fixed surface area involves balancing the dimensions to achieve desired proportions. By utilizing the mathematical relationships derived here, engineers can make informed decisions that optimize material usage and meet functional specifications.

Frequently Asked Questions (FAQs)

Q1: How does changing the surface area S affect the height-to-diameter ratio?

A: Increasing S increases $\frac{h}{d}$, leading to taller, narrower cans for the same radius, whereas decreasing S results in shorter, wider cans.

Q2: Is there an optimal radius for maximum volume given fixed surface area?

A: Yes. For fixed S, the volume $V = \pi r^2 h$ can be optimized with respect to r. The maximum volume occurs when the surface area is partitioned equally between the lateral surface and the ends, which can be derived through calculus.

Q3: Can this analysis be extended to open cylinders or other shapes?

A: Yes. Similar principles apply; however, the formulas for surface area and volume differ depending on the shape, requiring tailored derivations.

References

- Stewart, J. (2015

Frequently Asked Questions

How do I find the ratio of the height to the diameter of a closed cylindrical can with a fixed surface area?

You can set up the surface area formula for a cylinder, express height in terms of diameter, and then simplify to find the ratio of height to diameter.

What is the formula for the surface area of a closed cylinder?

The surface area $S = 2\pi r(h + r)$, where r is the radius and h is the height of the cylinder.

How do I express the height in terms of the diameter for the cylinder?

Since diameter $d = 2r$, you can write radius $r = d/2$, and then express height h in terms of d using the surface area equation.

If the surface area S is fixed, how does changing the height affect the radius and diameter?

For a fixed S, increasing the height h typically decreases the radius r, and vice versa, maintaining the surface area constant.

Is the ratio of height to diameter constant for a fixed surface area?

No, the ratio varies depending on the specific dimensions; you need to derive it mathematically using the surface area constraint.

Can you provide the step-by-step process to find the height-to-diameter ratio?

Yes. First, write the surface area formula in terms of d and h, then express h in terms of d using the fixed S, and finally find the ratio h/d.

What assumptions are made in deriving the ratio of height to diameter?

The main assumption is that the can is perfectly cylindrical, and the surface area S remains constant while dimensions change.

How does the fixed surface area influence the optimal dimensions of the cylinder?

It constrains the dimensions such that any change in height or diameter must compensate with the other to keep the surface area constant, affecting their ratio.

Additional Resources

A Closed Cylindrical Can Has A Fixed Surface Area S. Find The Ratio Of Its Height To The Diameter Of

Understanding the geometric properties of cylindrical cans is essential in various fields, from manufacturing and packaging to mathematics and engineering design. When dealing with a closed cylindrical can that has a fixed surface area S, one of the intriguing questions is determining the optimal shape or the ratio of its height to diameter that maximizes volume or minimizes material usage. This problem combines concepts of surface area, volume optimization, and algebraic manipulation, making it a rich topic for both theoretical and practical exploration.

Introduction to the Problem

The problem states that a closed cylindrical can has a fixed surface area S. The goal is to find the ratio of its height to its diameter, i.e., $\frac{h}{d}$, where h is the height of the cylinder, and d is the diameter of its base. This type of problem is common in optimization tasks, where constraints (fixed surface area) lead to the determination of an optimal ratio for specific dimensions.

In real-world applications, such optimization helps in designing cans that maximize volume (for product capacity) while minimizing material costs—an economical and efficient approach. The problem thus involves understanding the relationships between surface area, volume, and the dimensions of the cylinder.

Mathematical Formulation

To analyze this, let's define:

- r = radius of the base of the cylinder
- $d = 2r$ = diameter of the base
- h = height of the cylinder

The surface area S of a closed cylinder includes the lateral surface area and the area of the two circular bases:

$$
S = 2\pi r h + 2\pi r^2
$$

Given that S is fixed:

$$
2\pi r h + 2\pi r^2 = S
$$

Our goal is to find the ratio $\frac{h}{d}$, which can be rewritten as:

$$
\frac{h}{d} = \frac{h}{2r}
$$

To proceed, we'll express h in terms of r, then analyze the ratio.

Expressing h in terms of r

From the surface area equation:

$$
2\pi r h + 2\pi r^2 = S
$$

Solve for h:

$$
2\pi r h = S - 2\pi r^2
$$

$$
h = \frac{S - 2\pi r^2}{2\pi r}
$$

Simplify:

$$
h = \frac{S}{2\pi r} - r
$$

Now, the ratio $\frac{h}{d}$:

$$
\frac{h}{d} = \frac{h}{2r} = \frac{\frac{S}{2\pi r} - r}{2r} = \frac{S}{4\pi r^2} - \frac{r}{2r} = \frac{S}{4\pi r^2} - \frac{1}{2}
$$

Thus, the ratio $\frac{h}{d}$ in terms of r:

$$
\boxed{
\frac{h}{d} = \frac{S}{4\pi r^2} - \frac{1}{2}
}
$$

Optimizing the Volume with Respect to Dimensions

While the problem asks only for the ratio when the surface area is fixed, often the goal is to determine the ratio that maximizes volume V:

$$
V = \pi r^2 h
$$

Express V using the earlier expression for h:

$$
V = \pi r^2 \left(\frac{S}{2\pi r} - r \right) = \pi r^2 \left(\frac{S}{2\pi r} \right) - \pi r^3
$$

Simplify:

$$
V = r \frac{S}{2} - \pi r^3
$$

To find the value of r that maximizes V, differentiate with respect to r:

$$
\frac{dV}{dr} = \frac{S}{2} - 3 \pi r^2
$$

Set the derivative to zero for critical points:

$$
\frac{S}{2} - 3 \pi r^2 = 0
$$

$$
3 \pi r^2 = \frac{S}{2}
$$

$$
r^2 = \frac{S}{6\pi}
$$

$$
r = \sqrt{\frac{S}{6\pi}}
$$

Now, substitute back into the expression for h:

$$
h = \frac{S}{2\pi r} - r
$$

$$
h = \frac{S}{2\pi \sqrt{\frac{S}{6\pi}}} - \sqrt{\frac{S}{6\pi}}
$$

Simplify numerator:

$$
h = \frac{S}{2\pi} \times \frac{1}{\sqrt{\frac{S}{6\pi}}} - \sqrt{\frac{S}{6\pi}}
$$

Note that:

$$
\sqrt{\frac{S}{6\pi}} = \frac{\sqrt{S}}{\sqrt{6\pi}}
$$

So,

$$
h = \frac{S}{2\pi} \times \frac{\sqrt{6\pi}}{\sqrt{S}} - \frac{\sqrt{S}}{\sqrt{6\pi}}
$$

Simplify each term:

$$
h = \frac{\sqrt{6\pi} \times S}{2\pi \sqrt{S}} - \frac{\sqrt{S}}{\sqrt{6\pi}}
$$

$$
h = \frac{\sqrt{6\pi} \times \sqrt{S}}{2\pi} - \frac{\sqrt{S}}{\sqrt{6\pi}}
$$

Express both terms with common structure:

$$
h = \sqrt{S} \left(\frac{\sqrt{6\pi}}{2\pi} - \frac{1}{\sqrt{6\pi}} \right)
$$

To combine:

$$
h = \sqrt{S} \left(\frac{\sqrt{6\pi}}{2\pi} - \frac{1}{\sqrt{6\pi}} \right)
$$

Express both over common denominator 2π:

$$
\frac{\sqrt{6\pi}}{2\pi} - \frac{1}{\sqrt{6\pi}} = \frac{\sqrt{6\pi}}{2\pi} - \frac{\sqrt{6\pi}}{6\pi}
$$

Because:

$$
\frac{1}{\sqrt{6\pi}} = \frac{\sqrt{6\pi}}{6\pi}
$$

Now, subtract:

$$
\frac{\sqrt{6\pi}}{2\pi} - \frac{\sqrt{6\pi}}{6\pi} = \sqrt{6\pi} \left(\frac{1}{2\pi} - \frac{1}{6\pi} \right) = \sqrt{6\pi} \left(\frac{3 - 1}{6\pi} \right) = \sqrt{6\pi} \times \frac{2}{6\pi} = \sqrt{6\pi} \times \frac{1}{3\pi}
$$

Simplify:

$$
= \frac{\sqrt{6\pi}}{3\pi}
$$

\]

Thus,

\[
h = \sqrt{S} \times \frac{\sqrt{6\pi}}{3\pi}
\]

Recall that \(r = \sqrt{\frac{S}{6\pi}} \), then:

\[
h = \frac{\sqrt{6\pi} \times \sqrt{S}}{3\pi}
\]

Now, compute the ratio \(\frac{h}{d} \):

\[
d = 2r = 2 \sqrt{\frac{S}{6\pi}}
\]

Therefore,

\[
\frac{h}{d} = \frac{h}{2r} = \frac{\frac{\sqrt{6\pi} \sqrt{S}}{3\pi}}{2 \sqrt{\frac{S}{6\pi}}}
\]

Express the denominator:

\[
2 \sqrt{\frac{S}{6\pi}} = 2 \times \frac{\sqrt{S}}{\sqrt{6\pi}} = \frac{2 \sqrt{S}}{\sqrt{6\pi}}
\]

Now, write the ratio:

\[
\frac{h}{d} = \frac{\frac{\sqrt{6\pi} \sqrt{S}}{3\pi}}{\frac{2 \sqrt{S}}{\sqrt{6\pi}}} = \left(\frac{\sqrt{6\pi} \sqrt{S}}{3\pi} \right) \

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