A Consumer Has Utility Function U(x,y)=x A Y 1a Where 00 And Y>0 For Interior Solutions. (a) Find

A Consumer Has Utility Function $U(x,y)=x^a y^{1-a}$ where $0 < a < 1$ and $y > 0$ for interior solutions. (a) Find

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Introduction

Understanding consumer behavior through utility functions is a fundamental aspect of microeconomics. The utility function $U(x,y) = x^a y^{1-a}$, where $0 < a < 1$ and $y > 0$, represents a typical form of Cobb-Douglas utility. This function models consumer preferences over two goods, x and y, capturing the idea of diminishing marginal returns and the trade-offs consumers face when allocating their income.

In this article, we will delve into the derivation of the consumer's optimal consumption bundle given the utility function $U(x,y) = x^a y^{1-a}$. We will explore the conditions for interior solutions—where both goods are positive—and provide a detailed step-by-step process to find the optimal quantities of x and y that maximize utility subject to a budget constraint.

Understanding the Utility Function

The Form of the Utility Function

The utility function $U(x,y) = x^a y^{1-a}$ is a classic example of a Cobb-Douglas utility function. Key features include:

- Proportional preferences: The exponents a and $1-a$ indicate the relative importance of goods x and y in the consumer's preferences.
- Constant Elasticity of Substitution (CES): The Cobb-Douglas form implies a specific constant elasticity of substitution between goods.
- Positive marginal utility: For all $x > 0$ and $y > 0$, the marginal utilities are positive, reflecting that more of each good increases utility.

Conditions for Interior Solutions

An interior solution occurs when the consumer consumes positive quantities of both goods, i.e., $x > 0$ and $y > 0$. To ensure such solutions, the utility function should be smooth and strictly increasing in both arguments, and the consumer's budget constraint must allow for positive consumption bundles.

Setting Up the Optimization Problem

The Consumer's Budget Constraint

Let:

- I = consumer's income
- p_x = price of good x
- p_y = price of good y

The budget constraint is:

$$p_x x + p_y y = I$$

The consumer aims to maximize their utility:

$$\text{Maximize } U(x,y) = x^a y^{1-a}$$

Subject to:

$$p_x x + p_y y = I$$

And the non-negativity constraints:

$$x \geq 0, \quad y \geq 0$$

Solving the Optimization Problem

Step 1: Formulate the Lagrangian

Introduce a Lagrangian multiplier λ for the budget constraint:

$$\mathcal{L}(x,y,\lambda) = x^a y^{1-a} + \lambda (I - p_x x - p_y y)$$

Step 2: Find the First-Order Conditions (FOCs)

Differentiating $\mathcal{L}$ with respect to x, y, and λ:

1. $\frac{\partial \mathcal{L}}{\partial x} = a x^{a-1} y^{1-a} - \lambda p_x = 0$

2. $\frac{\partial \mathcal{L}}{\partial y} = (1-a) x^{a} y^{-a} - \lambda p_y = 0$

3. $\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_x x - p_y y = 0$

Step 3: Derive the Marginal Rate of Substitution (MRS)

Dividing the first FOC by the second:

$$
\frac{a x^{a-1} y^{1-a}}{(1-a) x^{a} y^{-a}} = \frac{\lambda p_x}{\lambda p_y}
$$

Simplify numerator and denominator:

$$
\frac{a x^{a-1} y^{1-a}}{(1-a) x^{a} y^{-a}} = \frac{a}{1-a} \times \frac{x^{a-1}}{x^{a}} \times \frac{y^{1-a}}{y^{-a}}
$$

$$
= \frac{a}{1-a} \times x^{-1} \times y^{1-a + a} = \frac{a}{1-a} \times \frac{y}{x}
$$

Since λ cancels out, the ratio simplifies to:

$$
\frac{a}{1-a} \times \frac{y}{x} = \frac{p_x}{p_y}
$$

Step 4: Solve for the Optimal Ratio of Goods

Rearranged:

$$
\frac{y}{x} = \frac{(1-a)}{a} \times \frac{p_x}{p_y}
$$

or equivalently:

$$
y = x \times \frac{(1-a)}{a} \times \frac{p_x}{p_y}
$$

Deriving the Optimal Quantities

Step 1: Plug into the Budget Constraint

Substitute y into the budget constraint:

$$
p_x x + p_y y = I
$$

$$
p_x x + p_y \left(x \times \frac{(1-a)}{a} \times \frac{p_x}{p_y} \right) = I
$$

Simplify:

$$
p_x x + p_x x \times \frac{(1-a)}{a} = I
$$

$$
p_x x \left(1 + \frac{(1-a)}{a} \right) = I
$$

Express the sum inside parentheses:

$$
1 + \frac{1-a}{a} = \frac{a}{a} + \frac{1-a}{a} = \frac{a + 1-a}{a} = \frac{1}{a}
$$

Therefore:

$$
p_x x \times \frac{1}{a} = I
$$

Solve for x:

$$
x^ = \frac{a I}{p_x}
$$

Step 2: Find y in terms of x

Recall:

$$
y = x \times \frac{(1-a)}{a} \times \frac{p_x}{p_y}
$$

Substitute $x^ = \frac{a I}{p_x}$:

$$
y^ = \frac{a I}{p_x} \times \frac{(1-a)}{a} \times \frac{p_x}{p_y} = I \times \frac{(1-a)}{p_y}
$$

Final Optimal Consumption Bundle

$$
\boxed{
\begin{aligned}
x^ &= \frac{a I}{p_x} \\
y^ &= \frac{(1-a) I}{p_y}
\end{aligned}
}
$$

This solution indicates that the consumer allocates their income proportionally according to the exponents in the utility function, confirming the intuitive result that in Cobb-Douglas utility maximization, the expenditure shares are constant and equal to the exponents.

Implications and Insights

1. Proportional Expenditure Shares

- The consumer spends a fixed proportion a of income on good x and $1 - a$ on good y.
- The expenditure on each good:

$$
\text{Expenditure on } x = p_x x^ = a I
$$
$$
\text{Expenditure on } y = p_y y^ = (1 - a) I
$$

2. Impact of Prices and Income

- An increase in income I leads to proportional increases in consumption.
- Changes in prices p_x and p_y affect consumption inversely, consistent with basic economic intuition.

3. Interior Solution Conditions

- Since $a \in (0,1)$, both $x^$ and $y^$ are strictly positive for positive income and prices.
- The solution holds as long as prices and income satisfy the budget constraint.

Extensions and Applications

A. Consumer Choice Theory

The derived solution exemplifies how consumers allocate their income optimally to maximize utility, a key concept in consumer choice theory.

B. Market Analysis

Understanding these consumption patterns helps firms and policymakers predict demand responses to price changes and income variations.

C. Welfare Economics

The explicit form of the utility maximization problem aids in evaluating consumer welfare and designing efficient markets.

Conclusion

The utility function $U(x,y) = x^a y^{1-a}$ captures essential features of consumer preferences with a straightforward optimization process. By applying the method of Lagrange multipliers, we find that the consumer's optimal consumption bundle allocates income proportionally according to the exponents in the utility function. The optimal quantities are:

\[
x^ = \frac{a I}{p_x}
\]
\[
y^ = \frac{(1−a) I}{

Frequently Asked Questions

What is the utility function U(x,y) = x^a y^b, and what do the parameters a and b represent?

The utility function U(x,y) = x^a y^b is a Cobb-Douglas utility function where x and y are quantities of two goods, and a and b are positive parameters representing the elasticities of utility with respect to each good. They indicate how changes in each good affect overall utility.

How do you find the interior solution for the utility maximization problem with the utility function U(x,y) = x^a y^b?

To find the interior solution, set up the Lagrangian with the budget constraint and solve the first-order conditions for x and y. Typically, the solution involves equating the ratio of marginal utilities to the ratio of prices and solving for x and y, ensuring both are positive.

What conditions must hold for the interior solution to be optimal in this utility maximization problem?

The interior solution requires that the marginal rate of substitution (MRS) equals the price ratio, and that the consumer's budget constraint is satisfied with positive quantities of both goods. Additionally, the parameters a and b should be positive to ensure interior solutions.

How are the demand functions for x and y derived from the utility function $U(x,y) = x^a y^b$?

Demand functions are derived by solving the utility maximization problem using the budget constraint. For Cobb-Douglas functions, the demand functions are proportional to income divided by the price, specifically x = (a / (a + b)) (I / Px) and y = (b / (a + b)) (I / Py), where I is income.

What is the significance of the parameters a and b being positive in the utility function?

Positive parameters a and b indicate that both goods contribute positively to utility, and consumers derive satisfaction from consuming more of each good. It also ensures the existence of interior solutions where both goods are consumed in positive quantities.

How does the assumption Y > 0 influence the utility maximization problem?

Assuming Y > 0 ensures that the quantity of good y is positive, which is necessary for the interior solution. It implies the consumer consumes some positive amount of y, avoiding corner solutions where y could be zero.

Can you explain the role of the budget constraint in solving for the utility-maximizing quantities for x and y?

The budget constraint, typically Pxx + Pyy = I, limits the consumer's choice set. Solving the utility maximization problem involves finding the combination of x and y that maximizes utility while satisfying this constraint, leading to demand functions that depend on prices and income.

What are the typical steps involved in solving for the interior solution of $U(x,y) = x^a y^b$?

The steps include: 1) Setting up the Lagrangian with the utility function and budget constraint, 2) Calculating first-order conditions (partial derivatives), 3) Equating the ratio of marginal utilities to the price ratio, 4) Solving these equations simultaneously with the budget constraint to find x and y, ensuring both are positive for an interior solution.

Additional Resources

A Consumer Has Utility Function $U(x,y)=x^a y^{1-a}$ Where $0 < a < 1$ And $y > 0$ For Interior Solutions. (a) Find

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Introduction: Understanding Consumer Utility and the Significance of the Function

In microeconomic theory, consumer preferences are often modeled through utility functions—mathematical representations that quantify the satisfaction or happiness derived from consuming different combinations of goods. Among the myriad of utility functions, the Cobb-Douglas form stands out for its analytical simplicity and practical relevance. When a consumer's utility is expressed as $U(x, y) = x^a y^{1-a}$, where $0 < a < 1$ and $y > 0$, it encapsulates a set of preferences that assume a constant proportionate trade-off between two goods, x and y.

This article explores the intricacies of such a utility function, particularly focusing on the scenario of interior solutions—situations where the consumer chooses a positive amount of both goods. We will dissect the problem of finding the consumer's optimal consumption bundle, given their utility function, and elucidate the underlying economic principles that govern this optimization. By the end, readers will gain a comprehensive understanding of how to analyze and solve utility maximization problems involving Cobb-Douglas preferences.

The Structure of the Utility Function: Cobb-Douglas Preferences

What is the Utility Function $U(x, y) = x^a y^{1-a}$?

The utility function under consideration is a classic form known as the Cobb-Douglas utility function:

$$U(x, y) = x^a y^{1 - a}$$

where:
- x and y are quantities of goods consumed,
- a $(0 < a < 1)$ is the parameter reflecting the consumer's relative preference for good x versus good y.

This function possesses several key properties:
- Diminishing Marginal Utility: The additional satisfaction gained from consuming an extra unit of a good decreases as consumption increases.
- Constant Elasticity of Substitution: The rate at which the consumer is

willing to substitute one good for another remains constant, which simplifies analysis.
- Preference for Diversification: Because of the multiplicative nature, the consumer prefers a balanced bundle rather than extreme consumption.

Why assume interior solutions?

Interior solutions imply the consumer consumes positive quantities of both goods, avoiding corner solutions where consumption of one good drops to zero. For such solutions to exist, certain conditions related to the consumer's budget and preferences must be satisfied, which we will explore in subsequent sections.

The Consumer’s Optimization Problem: Setting Up the Framework

Budget Constraint

The consumer faces a budget constraint, which limits the total expenditure:

$$ p_x x + p_y y = I $$

where:
- p_x and p_y are the prices of goods x and y respectively,
- I is the consumer's income or budget.

The goal is to choose x and y to maximize utility while satisfying this constraint.

The Optimization Objective

Mathematically, the problem is formulated as:

$$
\begin{aligned}
& \text{Maximize} \quad U(x, y) = x^a y^{1 - a} \\
& \text{Subject to} \quad p_x x + p_y y = I, \quad x \geq 0, y \geq 0
\end{aligned}
$$

Given the utility function's form, solving this involves applying methods such as Lagrangian multipliers, which facilitate handling the constraint.

Deriving the Optimal Consumption Bundle

Step 1: Constructing the Lagrangian

Set up the Lagrangian function:

$$
\mathcal{L} = x^a y^{1 - a} + \lambda (I - p_x x - p_y y)
$$

where λ is the Lagrange multiplier associated with the budget constraint.

Step 2: Calculating Partial Derivatives

Find the first-order conditions (FOCs):

$$
\frac{\partial \mathcal{L}}{\partial x} = a x^{a-1} y^{1 - a} - \lambda p_x = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial y} = (1 - a) x^{a} y^{-a} - \lambda p_y = 0
$$

$$
\frac{\partial \mathcal{L}}{\partial \lambda} = I - p_x x - p_y y = 0
$$

Step 3: Solving the FOCs

Dividing the first FOC by the second:

$$
\frac{a x^{a-1} y^{1 - a}}{(1 - a) x^{a} y^{-a}} = \frac{\lambda p_x}{\lambda p_y}
$$

Simplify numerator and denominator:

$$
\frac{a y}{(1 - a) x} = \frac{p_x}{p_y}
$$

Rearranged:

$$
\frac{a}{1 - a} \cdot \frac{y}{x} = \frac{p_x}{p_y}
$$

Hence, the ratio of consumption:

$$
\frac{y}{x} = \frac{(1 - a)}{a} \cdot \frac{p_x}{p_y}
$$

This key result indicates that at the optimum, the consumer allocates expenditure proportionally, taking into account their preferences and relative prices.

Step 4: Expressing y in terms of x

From the above:

$$
y = \frac{(1 - a)}{a} \cdot \frac{p_x}{p_y} \cdot x
$$

Step 5: Applying the Budget Constraint

Substitute y into the budget constraint:

$$
p_x x + p_y \left[\frac{(1 - a)}{a} \cdot \frac{p_x}{p_y} \cdot x \right] = I
$$

Simplify:

$$
p_x x + \frac{(1 - a)}{a} p_x x = I
$$
$$
p_x x \left[1 + \frac{(1 - a)}{a} \right] = I
$$

Calculate the bracket:

$$
1 + \frac{(1 - a)}{a} = \frac{a + (1 - a)}{a} = \frac{1}{a}
$$

Thus:

$$
p_x x \cdot \frac{1}{a} = I
$$
$$
x = \frac{a I}{p_x}
$$

Similarly, substitute back for y:

$$
y = \frac{(1 - a)}{a} \cdot \frac{p_x}{p_y} \cdot \frac{a I}{p_x} = \frac{(1 - a) I}{p_y}
$$

Final Results: The Consumer’s Optimal Consumption Bundle

The optimal amounts of goods \(x \) and \(y \) are:

\[
\boxed{
\begin{aligned}
x^ &= \frac{a I}{p_x} \\
y^ &= \frac{(1 - a) I}{p_y}
\end{aligned}
}
\]

These results have profound implications:

- The consumer allocates their income proportionally to their preference parameter \(a \) and \(1 - a \).
- The quantities depend on the prices and income, aligning with the intuitive idea that higher prices reduce demand, while higher income increases consumption.

The Nature of Interior Solutions: Conditions for Positivity

The derived solutions assume interior solutions, meaning both \(x^ > 0 \) and \(y^ > 0 \). For this to hold:

- \(a \) must be strictly between 0 and 1 (which is given).
- Income \(I \) must be positive.
- Prices \(p_x \) and \(p_y \) must be positive.

Under these conditions, the consumer’s optimal choice involves consuming both goods in positive quantities, maintaining a balanced consumption bundle aligned with their preferences.

Economic Intuition and Broader Implications

Why does the consumer allocate income proportionally?

This proportionality stems from the Cobb-Douglas form's property of constant expenditure shares. Specifically, the consumer spends a fixed fraction \(a \) of income on good \(x \) and \(1 - a \) on good \(y \). Unlike other utility functions where demand may vary unpredictably, the Cobb-Douglas preferences ensure stable expenditure proportions regardless of income changes, holding prices constant.

Impacts of Price Changes

- An increase in \(p_x \) will decrease \(x^ \), as expected.
- Similarly, a rise in \(p_y \) reduces \(y^ \).
- These responses are linear, making Cobb-Douglas preferences particularly tractable for economic modeling.

Practical Applications

Understanding such utility maximization is crucial in various contexts:

- Consumer Behavior Analysis: Estimating consumer responses to price changes.
- Market Demand Forecasting: Predicting how demand shifts with economic variables.
- Policy Impact Studies: Assessing how taxation or subsidies influence consumption patterns.

Extending the Model: Beyond Interior Solutions

While the current analysis assumes interior solutions, real-world scenarios

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