A Factorization A = PDP^-1 Is Not Unique. For A = [9 -12 2 1], One Factorization Is P = [1 -2 1 -3],

A Factorization A = PDP^-1 Is Not Unique. For A = [9 -12; 2 1], One Factorization Is P = [1 -2; 1 -3]

Introduction to Matrix Diagonalization and Similarity Transformations

Matrix diagonalization is a fundamental concept in linear algebra, allowing us to simplify complex matrix computations by transforming a matrix into a diagonal form. When a matrix A can be expressed as $A = P D P^{-1}$, where:

- D is a diagonal matrix containing the eigenvalues of A,
- P is an invertible matrix composed of the eigenvectors of A,

we say that A is diagonalizable. This form is especially useful because powers of A and functions of A become straightforward to compute via the diagonal matrix D.

However, an important aspect of this decomposition is its non-uniqueness. Multiple different matrices P and D can satisfy the equation for the same matrix A, leading to various factorizations. This article explores why the factorization $A = P D P^{-1}$ is not unique, illustrated through the example of the matrix $A = \begin{bmatrix} 9 & -12 \\ 2 & 1 \end{bmatrix}$ with a specific factorization $P = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}$.

Understanding the Non-Uniqueness of Similarity Transformations

Eigenvalues and Eigenvectors: The Core of Diagonalization

At the heart of diagonalization is the concept of eigenvalues and eigenvectors:

- Eigenvalues (λ): Scalars such that $A \mathbf{v} = \lambda \mathbf{v}$
- Eigenvectors ($\mathbf{v}$): Non-zero vectors satisfying the above relation.

The matrix D contains the eigenvalues along its diagonal, and the columns of P are the corresponding eigenvectors.

Key Point: The eigenvalues of A are unique (up to ordering), but the eigenvectors are not. Any scalar multiple of an eigenvector is also an eigenvector, leading to multiple possible eigenvector choices.

Why Is the Factorization Not Unique?

The non-uniqueness of the factorization $A = P D P^{-1}$ hinges on the following:

- Eigenvector Scaling: For each eigenvector, any non-zero scalar multiple is also an eigenvector. This allows for infinitely many choices of eigenvectors corresponding to the same eigenvalue.
- Choice of Basis in Eigenvector Space: Different bases spanning the eigenspaces lead to different matrices P, even though they diagonalize A identically.
- Permutation of Eigenvalues and Eigenvectors: The order in which eigenvalues and eigenvectors are placed in D and P can vary, resulting in different factorizations.

In essence: The matrices P and D are not uniquely determined by A. Instead, they depend on the particular choice of eigenvectors and their scaling.

Explicit Example: The Matrix $A = \begin{bmatrix} 9 & -12 \\ 2 & 1 \end{bmatrix}$

Eigenvalues of A

Let's find the eigenvalues of A:

$$
A = \begin{bmatrix} 9 & -12 \\ 2 & 1 \end{bmatrix}
$$

Eigenvalues satisfy the characteristic equation:

$$
\det(A - \lambda I) = 0
$$

Calculating:

$$
\det \begin{bmatrix} 9 - \lambda & -12 \\ 2 & 1 - \lambda \end{bmatrix} = (9 - \lambda)(1 - \lambda) - (-12)(2) = 0
$$

$$
(9 - \lambda)(1 - \lambda) + 24 = 0
$$

Expanding:

$$
(9 \times 1) - 9 \lambda - \lambda + \lambda^2 + 24 = 0
$$

$$
9 - 9 \lambda - \lambda + \lambda^2 + 24 = 0
$$

$$
(\lambda^2 - 10 \lambda + 33) = 0
$$

Solving the quadratic:

$$
\lambda^2 - 10 \lambda + 33 = 0
$$

Discriminant:

$$
\Delta = 100 - 4 \times 1 \times 33 = 100 - 132 = -32
$$

Since the discriminant is negative, the eigenvalues are complex conjugates:

$$
\lambda_{1,2} = \frac{10 \pm \sqrt{-32}}{2} = \frac{10 \pm i \sqrt{32}}{2} = 5 \pm i \sqrt{8} = 5 \pm 2 i \sqrt{2}
$$

Eigenvalues:

$$
\boxed{
\lambda_1 = 5 + 2 i \sqrt{2} \quad \text{and} \quad \lambda_2 = 5 - 2 i \sqrt{2}
}
$$

Eigenvectors and Multiple Factorizations

Suppose we find an eigenvector $\mathbf{v}_1$ associated with λ_1. Because eigenvectors are determined up to scalar multiplication, any non-zero scalar multiple of $\mathbf{v}_1$ is also an eigenvector.

Similarly, for λ_2, the eigenvector $\mathbf{v}_2$ can be scaled arbitrarily. Selecting different eigenvector bases leads to different matrices P, which still satisfy:

$$
A = P D P^{-1}
$$

but with different P and D.

The Given Factorization and Its Variability

Specific Example: $P = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}$

The example provided mentions one particular factorization:

$$

$$
A = P D P^{-1}
$$

with

$$
P = \begin{bmatrix} 1 & -2 \\ 1 & -3 \end{bmatrix}
$$

and D being a diagonal matrix with the eigenvalues.

Note: Without explicit eigenvalues, the focus is on understanding that this particular P is constructed from specific eigenvectors (or generalized eigenvectors), but alternative P' matrices can be constructed by choosing different eigenvectors or scaling factors.

Implications of Different Choices

- Scaling Eigenvectors: Multiplying an eigenvector by a non-zero scalar results in a different P.
- Changing Eigenvector Bases: Choosing a different basis for the eigenspace yields a different P.
- Permutation of Eigenvalues and Eigenvectors: Rearranging the order affects the structure of P and D.

Consequently: The factorization $A = P D P^{-1}$ is not unique; infinitely many factorizations exist corresponding to different eigenvector choices.

Mathematical Formalism of Non-Uniqueness

Eigenvector Scaling and the General Form of P

Suppose P is a matrix of eigenvectors:

$$
P = [\mathbf{v}_1 \quad \mathbf{v}_2]
$$

Any scaled version:

$$
P' = [k_1 \mathbf{v}_1 \quad k_2 \mathbf{v}_2]
$$

where $k_1, k_2 \neq 0$, will still diagonalize A:

$$
A = P' D' P'^{-1}
$$

with appropriately adjusted D'.

This demonstrates that the choice of eigenvectors directly influences the factorization.

Change of Basis in Eigenvector Space

More generally, if Q is an invertible 2×2 matrix, then:

$$
P' = P Q
$$

and

$$
A = P D P^{-1} = (P Q) (Q^{-1} D Q) (Q^{-1} P^{-1}) = P' D' P'^{-1}
$$

where

$$
D' = Q^{-1} D Q
$$

This shows that, through similarity transformations within the eigenspace, different factorizations are possible.

Conclusion: The Non-Uniqueness of the Diagonalization Factorization

The key takeaways are:

1. Eigenvalues are unique (up to ordering), but eigenvectors are not.
2. Scaling and basis choices in eigenvector spaces lead to multiple valid factorizations $A = P D P^{-1}$.
3. Any invertible change of basis within the eigenspaces produces a different P and

Frequently Asked Questions

Why is the factorization $A = P D P^{-1}$ not unique for a matrix A?

Because different choices of the eigenvector matrix P and corresponding diagonal matrix D can produce the same matrix A, especially when eigenvalues have multiplicities or when eigenvectors are not uniquely determined, leading to multiple valid factorizations.

Given $A = [9\ -12;\ 2\ \ 1]$ and one factorization $P = [1\ -2;\ 1\ -3]$, how can we verify this factorization?

You can verify by computing $P D P^{-1}$ with the appropriate D (diagonal matrix of eigenvalues) and checking if the product equals A. Alternatively, confirm that P contains eigenvectors of A and D contains the corresponding eigenvalues.

What role does eigenvalue multiplicity play in the non-uniqueness of the $A = P D P^{-1}$ factorization?

Eigenvalue multiplicity allows for different choices of eigenvectors within the eigenspaces, leading to multiple valid P matrices and thus multiple factorizations, making the decomposition non-unique.

Can the factorization $A = P D P^{-1}$ be unique if all eigenvalues are distinct?

Yes, if all eigenvalues are distinct, the eigenvectors are uniquely determined (up to scaling), making the eigen-decomposition (and thus the factorization) essentially unique, aside from the ordering and scaling of eigenvectors.

What information is needed to determine a specific factorization $A = P D P^{-1}$?

You need the eigenvalues of A to form D and a set of linearly independent eigenvectors to form P. The choice of eigenvectors determines the specific P and D, and different choices lead to different factorizations.

In the given example, how might different choices of P affect the factorization $A = P D P^{-1}$?

Different sets of eigenvectors can lead to different P matrices, which in turn produce different D matrices when diagonalized, illustrating the non-uniqueness of the factorization. The provided P is just one valid choice.

Additional Resources

Factorization
Unraveling the Nuances of the $A = PDP^{-1}$ Decomposition and Its Non-Uniqueness

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Introduction

Matrix factorization is a cornerstone of linear algebra, serving as a powerful tool for understanding the structure of matrices, solving systems of equations, and facilitating various computational algorithms. Among the myriad forms of matrix decomposition, the similarity transformation expressed as

$$
A = P D P^{-1}
$$

stands out due to its foundational role in diagonalization, spectral theory, and many applied disciplines like engineering, physics, and computer science.

However, a critical aspect often overlooked is the non-uniqueness of such factorizations. While at first glance, the process might seem straightforward—finding a suitable invertible matrix P and a diagonal matrix D such that $A = P D P^{-1}$—the reality is more nuanced. Multiple different pairs (P, D) can produce the same matrix A, revealing a layer of richness and complexity in matrix theory.

This article delves into this fascinating topic, exploring why the similarity factorization isn't unique, exemplified by the matrix

\[

$$
A = \begin{bmatrix}
9 & -12 \\
2 & 1
\end{bmatrix}
$$

and one such factorization involving

$$
P = \begin{bmatrix}
1 & -2 \\
1 & -3
\end{bmatrix}
$$

We will carefully analyze the concepts, work through the computations, and clarify why multiple factorizations exist, emphasizing the importance of understanding the underlying eigenstructure.

The Foundations of Similarity Transformations

What Is the $A = P D P^{-1}$ Factorization?

At its core, the expression

$$
A = P D P^{-1}
$$

is a diagonalization of matrix A, where:

- A is a square matrix.
- D is a diagonal matrix whose entries are the eigenvalues of A.
- P is an invertible matrix whose columns are the eigenvectors of A.

This factorization is central because it simplifies many matrix operations—especially powers and functions of matrices—by converting to a basis where the matrix acts as a mere scaling.

Eigenvalues and Eigenvectors: The Cornerstones

To understand the factorization, recall that:

- Eigenvalues λ_i satisfy $A \mathbf{v}_i = \lambda_i \mathbf{v}_i$.
- Eigenvectors $\mathbf{v}_i$ are vectors associated with λ_i.

The matrix P is constructed from these eigenvectors, and D from the eigenvalues.

Analyzing Our Specific Matrix A

Let's examine the matrix

$$
A = \begin{bmatrix}
9 & -12 \\
2 & 1
\end{bmatrix}
$$

\]

and explore its eigenstructure to understand possible factorizations.

Eigenvalues of A

The eigenvalues are found by solving the characteristic polynomial:

$$
\det(A - \lambda I) = 0
$$

Calculating:

$$
\det \begin{bmatrix}
9 - \lambda & -12 \\
2 & 1 - \lambda
\end{bmatrix}
= (9 - \lambda)(1 - \lambda) - (-12)(2) = 0
$$

Expanding:

$$
(9 - \lambda)(1 - \lambda) + 24 = 0
$$

$$
(9 \times 1 - 9 \lambda - \lambda + \lambda^2) + 24 = 0
$$

$$
(9 - 10 \lambda + \lambda^2) + 24 = 0
$$

$$
\lambda^2 - 10 \lambda + 33 = 0
$$

Using the quadratic formula:

$$
\lambda = \frac{10 \pm \sqrt{(-10)^2 - 4 \times 1 \times 33}}{2}
$$

$$
= \frac{10 \pm \sqrt{100 - 132}}{2}
$$

$$
= \frac{10 \pm \sqrt{-32}}{2}
$$

$$
= \frac{10 \pm i \sqrt{32}}{2}
$$

\[

$$
= 5 \pm i \sqrt{8}
$$

Since $\sqrt{8} = 2 \sqrt{2}$, the eigenvalues are:

$$
\boxed{
\lambda_{1,2} = 5 \pm 2 i \sqrt{2}
}
$$

These are complex conjugates, indicating that A is not diagonalizable over the real numbers but is diagonalizable over the complex field.

The Eigenvectors and the Non-Unique Factorization

Eigenvectors for A

Let's compute an eigenvector corresponding to $\lambda = 5 + 2 i \sqrt{2}$:

Solve:

$$
(A - \lambda I) \mathbf{v} = 0
$$

$$
\begin{bmatrix}
9 - \lambda & -12 \\
2 & 1 - \lambda
\end{bmatrix}
\begin{bmatrix}
x \\ y
\end{bmatrix}
= \mathbf{0}
$$

Plugging in $\lambda = 5 + 2 i \sqrt{2}$:

- $9 - \lambda = 9 - 5 - 2 i \sqrt{2} = 4 - 2 i \sqrt{2}$
- $1 - \lambda = 1 - 5 - 2 i \sqrt{2} = -4 - 2 i \sqrt{2}$

The system:

$$
(4 - 2 i \sqrt{2}) x - 12 y = 0
$$
$$
2 x + (-4 - 2 i \sqrt{2}) y = 0
$$

From the first:

$$
(4 - 2 i \sqrt{2}) x = 12 y \Rightarrow y = \frac{(4 - 2 i \sqrt{2})}{12} x
$$

Choose $x = 1$:

$$
y = \frac{4 - 2 i \sqrt{2}}{12} = \frac{1}{3} - \frac{i \sqrt{2}}{6}
$$

Thus, an eigenvector:

$$
\mathbf{v}_1 = \begin{bmatrix}
1 \\
\frac{1}{3} - \frac{i \sqrt{2}}{6}
\end{bmatrix}
$$

Similarly, for $\lambda = 5 - 2 i \sqrt{2}$, the eigenvector is the complex conjugate.

The Non-Uniqueness of the $A = P D P^{-1}$ Decomposition

Why Are Multiple Factorizations Possible?

The core reason for non-uniqueness is the freedom in choosing eigenvectors:

- Eigenvectors can be scaled arbitrarily (as long as they are non-zero).
- Eigenvectors associated with the same eigenvalue form a subspace, allowing for linear combinations.

Consequently, the matrix P, which is constructed from eigenvectors, can be modified by multiplying columns by non-zero scalars, and the corresponding diagonal matrix D adjusts accordingly, leading to different, yet valid, factorizations.

Transformations Within Eigenvector Space

Suppose A has an eigenvector $\mathbf{v}$, then:

$$
A \mathbf{v} = \lambda \mathbf{v}
$$

If we replace $\mathbf{v}$ with $\alpha \mathbf{v}$, where $\alpha \neq 0$, then:

$$
A (\alpha \mathbf{v}) = \alpha A \mathbf{v} = \alpha \lambda \mathbf{v} = \lambda (\alpha \mathbf{v})
$$

which still satisfies the eigenvalue equation.

Thus, the eigenvector basis is not unique; scaling factors provide alternative bases, leading to different matrices P.

The Specific Example with P:

Given the matrix

$$
P = \begin{bmatrix}
$$

1 & -2 \\
1 & -3
\end{bmatrix}
\]

and the fact that \(A = P D P^{-1}\), we can analyze whether this is a valid diagonalization.

Computing \(P^{-1}\)

Calculate the determinant:

\[
\det P = (1)(-3) - (-2)(1) = -3 + 2 = -1
\]

Thus,

\[
P^{-1} = \frac{1}{-1}
\begin{bmatrix}
-3 & 2 \\
-1 & 1
\end{bmatrix}
= \begin{bmatrix}
3 & -2 \\
1 & -1
\end{bmatrix}
\]

Validating the Factorization

To verify \(A = P D P^{-1}\), one needs the diagonal matrix \(D\). Since the eigenvalues are complex conjugates, and the original matrix is real, the diagonalization over the complex field involves complex eigenvalues and eigenvectors.

Suppose the eigenvalues are \(\lambda

A Factorization A Pdp 1 Is Not Unique For A 9 12 2 1 One Factorization Is P 1 2 1 3

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