A Fair Coin Is Tossed Three Times. Let X Be The Number Of Heads On The First Toss And Y Be The Total

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When analyzing simple probability scenarios, tossing a coin multiple times serves as a foundational example. In this article, we explore the scenario of tossing a fair coin three times, focusing on two random variables: \(X \), the number of heads on the first toss, and \(Y \), the total number of heads obtained across all three tosses. Understanding the relationship between \(X \) and \(Y \), as well as their probabilities, provides valuable insights into basic probability theory, joint distributions, and the concept of independence.

Introduction to the Problem

Suppose we have a fair coin, meaning the probability of landing heads (H) or tails (T) on any individual toss is equal, each with a probability of 0.5. When we toss this coin three times, there are a total of \(2^3 = 8 \) equally likely outcomes. These outcomes can be represented as ordered triples, such as (H, H, T), (T, H, H), etc.

In this context:

- \(X \) is a random variable indicating whether the first toss results in a head (value 1) or tails (value 0).
- \(Y \) is the total number of heads obtained in all three tosses, ranging from 0 (no heads at all) to 3 (heads on all three tosses).

By analyzing \(X \) and \(Y \), we can explore questions such as:

- What is the probability that the first toss is a head?
- What is the probability that the total number of heads is a certain value?
- How are \(X \) and \(Y \) related? Are they independent?
- What is the joint probability distribution of \(X \) and \(Y \)?

Let's delve deeper into these aspects.

Defining the Random Variables

The Random Variable \(X \)

\(X \) is a Bernoulli random variable representing the outcome of the first toss:

\[

$$
X = \begin{cases}
1, & \text{if the first toss is heads} \\
0, & \text{if the first toss is tails}
\end{cases}
$$

Since the coin is fair, the probability:

$$
P(X=1) = P(\text{first toss is heads}) = 0.5
$$
$$
P(X=0) = P(\text{first toss is tails}) = 0.5
$$

The Random Variable Y

Y counts the total number of heads across all three tosses:

$$
Y = H_1 + H_2 + H_3
$$

where H_i is 1 if the i^{th} toss is heads, and 0 otherwise. The possible values of Y are 0, 1, 2, or 3.

Sample Space and Outcomes

The sample space Ω consists of 8 equally likely outcomes:

Outcome	First Toss	Second Toss	Third Toss	Total Heads Y
1	H	H	H	3
2	H	H	T	2
3	H	T	H	2
4	H	T	T	1
5	T	H	H	2
6	T	H	T	1
7	T	T	H	1
8	T	T	T	0

Each outcome has probability $\frac{1}{8}$.

Calculating Probabilities of Interest

Probability that the First Toss Is Heads

Since the first toss is independent of the subsequent two, and the coin is fair:

$$
P(X=1) = 0.5
$$

Similarly,

$$
P(X=0) = 0.5
$$

Distribution of Y

Let's examine the distribution of the total number of heads Y:

- $Y=0$: Only outcome 8 (T, T, T), probability $\frac{1}{8}$.
- $Y=1$: Outcomes 4, 6, 7, each with probability $\frac{1}{8}$, total $\frac{3}{8}$.
- $Y=2$: Outcomes 2, 3, 5, each with probability $\frac{1}{8}$, total $\frac{3}{8}$.
- $Y=3$: Outcome 1, probability $\frac{1}{8}$.

So, the probability distribution of Y:

y	$P(Y=y)$
0	1/8
1	3/8
2	3/8
3	1/8

Joint Distribution of X and Y

To understand the relationship between X and Y, we examine the joint probabilities $P(X=x, Y=y)$.

- For $X=1$ (first toss heads), outcomes are 1, 2, 3, 4.
- For $X=0$ (first toss tails), outcomes are 5, 6, 7, 8.

Let's compute $P(X=x, Y=y)$:

X	Y	Outcomes	Count	Probability
1	3	1	1	$\frac{1}{8}$
1	2	2, 3	2	$\frac{2}{8} = \frac{1}{4}$
1	1	4	1	$\frac{1}{8}$
0	2	5	1	$\frac{1}{8}$
0	1	6, 7	2	$\frac{2}{8} = \frac{1}{4}$
0	0	8	1	$\frac{1}{8}$

From this, the joint probabilities are:

- $P(X=1, Y=3) = \frac{1}{8}$
- $P(X=1, Y=2) = \frac{1}{4}$
- $P(X=1, Y=1) = \frac{1}{8}$
- $P(X=0, Y=2) = \frac{1}{8}$
- $P(X=0, Y=1) = \frac{1}{4}$
- $P(X=0, Y=0) = \frac{1}{8}$

Analyzing the Relationship Between X and Y

Are X and Y Independent?

To determine if X and Y are independent, check whether:

$$
P(X=x, Y=y) = P(X=x) \times P(Y=y)
$$

for all x, y.

Let's verify for $X=1$, $Y=2$:

$$
P(X=1, Y=2) = \frac{1}{4}
$$
$$
P(X=1) = 0.5
$$
$$
P(Y=2) = \frac{3}{8}
$$
$$
P(X=1) \times P(Y=2) = 0.5 \times \frac{3}{8} = \frac{3}{16} = 0.1875
$$

But,

$$
P(X=1, Y=2) = 0.25
$$

Since $0.25 \neq 0.1875$, X and Y are not independent.

Conditional Probabilities

Calculate the probability that the total number of heads is y given the first toss:

$$
P(Y=y \mid X=1) = \frac{P(X=1, Y=y)}{P(X=1)}
$$

- For \(y=3 \):

\[
P(Y=3 \mid X=1) = \frac{\frac{1}{8}}{0.5} = \frac{1/8}{1/2} = \frac{1}{8}
\times 2 = \frac{1}{4}
\]

- For \(y=2 \):

\[
P(Y=2 \mid X=1) = \frac{\frac{1}{4

Frequently Asked Questions

What is the probability that the first toss results in a head given that the total number of heads in three tosses is at least one?

Since each toss is independent and fair, the probability that the first toss is a head given at least one head appears is 1/2.

How many possible outcomes are there when a fair coin is tossed three times?

There are 8 possible outcomes, since each toss has 2 possibilities, totaling 2^3 = 8 outcomes.

What is the probability that the total number of heads Y equals 2, and the first toss results in a head?

The probability that Y=2 and X=1 (first toss is head, total heads are 2) is 1/8, because there are 3 outcomes with exactly 2 heads, and in 2 of those, the first toss is a head.

If X is the number of heads on the first toss and Y is the total number of heads in three tosses, what is the probability that both X and Y are equal to 1?

The probability that the first toss is a head (X=1) and total heads are 1 (Y=1) is 1/8, since the only outcome is H T T.

How can the joint probability distribution of X and Y be described for this three-toss scenario?

Since X can be 0 or 1, and Y can range from X to 3, the joint probabilities are determined by the outcomes where the first toss is a head or tail, and the total number of heads accordingly, with each outcome having a probability of 1/8.

Additional Resources

A Fair Coin Is Tossed Three Times. Let X Be The Number Of Heads On The First Toss And Y Be The Total

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Introduction

Imagine flipping a fair coin three times in succession. Each flip has an equal chance of landing on heads or tails, making the process a classic example of a Bernoulli trial in probability theory. In this scenario, two random variables are of interest: X, which denotes whether the first toss results in heads, and Y, which counts the total number of heads obtained across all three tosses. Such a simple setup opens the door to exploring fundamental concepts in probability, including joint distributions, conditional probabilities, and expectations. This article aims to dissect this problem carefully, providing both the technical framework and an accessible explanation to readers with varying levels of familiarity with probability theory.

The Experiment Setup and Random Variables

Defining the Experiment

The experiment involves flipping a fair coin three times, with each flip independent of the others. Because the coin is fair, each outcome—heads (H) or tails (T)—has a probability of 0.5. The sample space for this experiment consists of 8 equally likely outcomes:

- H H H
- H H T
- H T H
- H T T
- T H H
- T H T
- T T H
- T T T

Random Variables X and Y

- X: Indicates whether the first flip is a head or not. It is a Bernoulli random variable where:

- X = 1 if the first toss is heads.
- X = 0 if the first toss is tails.

- Y: Counts the total number of heads across all three tosses. Since each flip can be either 0 or 1 head, Y can take values from 0 to 3.

Understanding the joint behavior of X and Y requires analyzing their probabilities for different combinations of values, which can inform us about the likelihood of various outcomes and their implications.

Joint Distribution of X and Y

Computing the Probabilities

The key to understanding the problem is to compute the joint probability distribution P(X = x, Y = y) for all possible values of x and y.

Since the coin is fair and flips are independent:

- The probability of any specific sequence of three flips is (1/2)^3 = 1/8.

To find P(X = x, Y = y), we analyze the sequences that satisfy these conditions:

- For X = 1: The first flip is heads.
- For X = 0: The first flip is tails.

Similarly, Y counts total heads.

Case 1: X = 1 (First flip is heads)

Sequences where the first flip is heads:

Sequence	First flip	Total heads (Y)	Probability
H H H	H	3	1/8
H H T	H	2	1/8
H T H	H	2	1/8
H T T	H	1	1/8

From these, the joint probabilities are:

- P(X=1, Y=3) = 1/8
- P(X=1, Y=2) = 2/8 (H H T and H T H)

- P(X=1, Y=1) = 1/8

Case 2: X = 0 (First flip is tails)

Sequences where the first flip is tails:

Sequence	First flip	Total heads (Y)	Probability
T H H	T	2	1/8
T H T	T	1	1/8
T T H	T	1	1/8
T T T	T	0	1/8

Corresponding joint probabilities:

- P(X=0, Y=2) = 1/8
- P(X=0, Y=1) = 2/8
- P(X=0, Y=0) = 1/8

Summary of Joint Distribution

X	Y	Probability
1	3	1/8
1	2	2/8 = 1/4
1	1	1/8
0	2	1/8
0	1	2/8 = 1/4
0	0	1/8

Total probability sums to 1, confirming the consistency of the distribution.

Marginal Distributions

Distribution of X

Since X depends solely on the first flip:

- P(X=1) = P(first flip is heads) = 0.5
- P(X=0) = P(first flip is tails) = 0.5

Distribution of Y

Calculating the marginal distribution of Y:

- P(Y=0): Only T T T sequence → 1/8
- P(Y=1): Sequences with exactly one head:

- H T T (first flip heads? No, first flip tails, so exclude)

- T H T

- T T H

Wait, the sequences where total heads = 1:

- H T T → first flip is H, total heads=1

- T H T → total heads=1

- T T H → total heads=1

But note that in the first sequence, first flip is H, so X=1, Y=1.

Similarly, for sequences with Y=2:

- H H T

- H T H

- T H H

Finally, Y=3: H H H only.

Hence, the marginal probabilities:

- P(Y=0) = 1/8

- P(Y=1) = 3/8

- P(Y=2) = 3/8

- P(Y=3) = 1/8

Conditional Probabilities and Independence

Is X independent of Y?

To examine independence, we compare P(X=x, Y=y) with P(X=x) P(Y=y).

For example, check P(X=1, Y=2):

- P(X=1, Y=2) = 1/4

- P(X=1) = 0.5

- P(Y=2) = 3/8 = 0.375

- P(X=1) P(Y=2) = 0.5 0.375 = 0.1875 = 3/16

Since 1/4 = 4/16 ≠ 3/16, the joint probability differs from the product of marginals, indicating X and Y are not independent.

Conditional distribution: Probability of Y given X

Let's compute P(Y=y | X=x):

- For X=1:

- P(Y=3 | X=1) = P(X=1, Y=3) / P(X=1) = (1/8) / 0.5 = (1/8) / (4/8) = 1/4

- P(Y=2 | X=1) = (1/4) / 0.5 = (2/8) / (4/8) = 1/2

- P(Y=1 | X=1) = (1/8) / 0.5 = 1/4

Similarly, for X=0:

- P(Y=2 | X=0) = (1/8) / 0.5 = 1/4

- P(Y=1 | X=0) = (2/8) / 0.5 = 1/2

- P(Y=0 | X=0) = (1/8) / 0.5 = 1/4

This reveals that the distribution of Y conditioned on the first flip's outcome is different, highlighting the dependence between X and Y.

Expected Values and Variance

Expected value of Y

Since Y counts total heads:

- E[Y] = sum over y of y P(Y=y)

Calculating:

E[Y] = 0(1/8) + 1(3/8) + 2(3/8) + 3(1/8)

= 0 + (3/8) + (6/8) + (3/8) = (3 + 6 + 3)/8 = 12/8 = 1.5

Expected value of X

X is Bernoulli with parameter p=0.5:

- E[X] = 1 0.5 + 0 0.5 = 0.5

Covariance between X and Y

Cov(X, Y) = E[XY] - E[X]E[Y]

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Calculate E[XY]:

- XY can be 0 or 1, depending on the values.

Observe that:

- XY = 1 only when X=1 and Y≥1.

- For sequences with X=1:

- Sequences:

- H H H: XY=1,
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