

A Random Sample Of Size 15 Is Taken From A Normally Distributed Population Revealed A Sample Mean Of

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When conducting statistical analysis, understanding the properties and implications of sample data drawn from a population is fundamental. Suppose a researcher collects a random sample of size 15 from a population that is normally distributed. After measurement and data collection, the sample's mean is calculated. This scenario raises important questions about the population parameters, the reliability of the sample mean, and the statistical inferences that can be made. Such a setup is common in many fields, including quality control, medical research, social sciences, and engineering, emphasizing the importance of grasping the underlying principles and techniques used in analyzing small samples from normal populations.

Understanding the Context of the Sample and Population

The Significance of Sample Size

- Sample size ($n=15$) is relatively small, which influences the choice of statistical methods.
- Small samples are more susceptible to variability, making precise estimation and inference more challenging.
- Despite this, with a normally distributed population, certain parametric tests remain valid even for

small samples.

The Normal Distribution Assumption

- The population is assumed to be normally distributed, which simplifies many aspects of analysis.
- Normality assumption allows the use of specific statistical tools, like the t-distribution, for inference about the population mean.
- Verifying normality is often recommended, especially for small samples, through graphical (Q-Q plots) or statistical tests (Shapiro-Wilk, Kolmogorov-Smirnov).

Sample Mean and Its Statistical Properties

Definition of the Sample Mean

- The sample mean ($\bar{x}$) is calculated by summing all observations and dividing by the sample size.
- It serves as a point estimator of the population mean (μ).

Expected Value and Variance of the Sample Mean

- The expected value of $\bar{x}$ is equal to the population mean μ , indicating its unbiased nature.
- The variance of $\bar{x}$ is σ^2 / n , where σ^2 is the population variance.
- The standard deviation of the sample mean, called the standard error, is $\sigma / \sqrt{n}$.

Implications for Small Samples

- With small n , the standard error is larger, reflecting more variability and less precision.

- Accurate estimation of σ becomes critical, often requiring the use of sample standard deviation s in practice.

Estimating Population Parameters from the Sample Data

Point Estimation of the Population Mean

- The sample mean ($\bar{x}$) provides a single best estimate of μ .

Confidence Intervals for the Population Mean

- When σ is unknown (common in practice), the t-distribution is used.
- The $(1 - \alpha)$ confidence interval is given by:

$$\bar{x} \pm t_{(1 - \alpha/2, n-1)} \times \frac{s}{\sqrt{n}}$$

where:

- s is the sample standard deviation,
- $t_{(1 - \alpha/2, n-1)}$ is the critical value from the t-distribution with $(n-1)$ degrees of freedom.

Interpreting Confidence Intervals

- A 95% confidence interval implies that if many samples are taken, approximately 95% of such intervals will contain μ .
- For small samples, the interval tends to be wider, reflecting increased uncertainty.

Hypothesis Testing About the Population Mean

Formulating Hypotheses

- Null hypothesis (H_0): $\mu = \mu_0$ (a specified value).
- Alternative hypothesis (H_A): $\mu \neq \mu_0$ (two-sided), or $\mu > \mu_0$ / $\mu < \mu_0$ (one-sided).

Test Statistic

- When σ is unknown and the population is normal:

$$t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}}$$

- The t-statistic follows a t-distribution with $(n - 1)$ degrees of freedom.

Decision Rules

- Compare the calculated t-value to the critical t-value for the chosen significance level α .
- If $|t|$ exceeds the critical value, reject H_0 ; otherwise, do not reject.

Implications of Small Sample Size for Statistical Inference

Potential Challenges

- Increased sampling variability affects the stability of estimates.
- Larger confidence intervals are necessary to maintain the same confidence level.
- The t-distribution's heavier tails account for this increased uncertainty.

Ensuring Validity of Inference

- Verify the normality assumption to ensure the validity of t-based methods.
- Use graphical tools and tests to assess normality, especially with small samples.

Alternative Approaches

- Non-parametric methods (e.g., Wilcoxon signed-rank test) when normality assumption is questionable.
- Bootstrapping techniques to estimate confidence intervals without relying heavily on distributional assumptions.

Practical Example and Application

Suppose the Sample Mean and Standard Deviation Are Known

- Let's assume:
- $\bar{x} = 50$,
- $s = 5$,
- $n = 15$,
- and we want to estimate μ with 95% confidence.

Calculating the Confidence Interval

- Degrees of freedom: $(15 - 1 = 14)$.
- Critical t-value for 95% confidence and 14 df: approximately 2.145.
- Standard error: $(s / \sqrt{n} = 5 / \sqrt{15} \approx 1.29)$.
- Confidence interval:

$$\begin{aligned} & 50 \pm 2.145 \times 1.29 \approx 50 \pm 2.77 \\ & \end{aligned}$$

- Lower bound: approximately 47.23
- Upper bound: approximately 52.77

Interpreting Results

- We are 95% confident that the true population mean (μ) lies between approximately 47.23 and 52.77.

Conclusion and Summary of Key Points

- The sample mean is an unbiased estimator of the population mean, especially under normality assumptions.
- With a sample size of 15, the t-distribution is used for constructing confidence intervals and hypothesis tests when the population standard deviation is unknown.
- Small samples require careful checks of assumptions and wider intervals to account for

increased variability.

- Understanding the relationship between sample statistics and population parameters is essential for making valid inferences.
- Practical application of these principles allows researchers to estimate population parameters reliably, even with limited data, provided assumptions are met and appropriate methods are used.

Overall, the process of sampling, estimation, and hypothesis testing forms the foundation of inferential statistics, enabling informed decision-making based on limited data from normally distributed populations.

Frequently Asked Questions

What does a sample mean reveal about a normally distributed population?

The sample mean provides an estimate of the population mean, indicating the central tendency of the data drawn from the population.

How can we use a sample of size 15 to perform hypothesis testing about the population mean?

By calculating the sample mean and using the t-distribution (since the sample size is small and the population standard deviation is unknown), we can perform a t-test to assess hypotheses about the population mean.

What is the significance of the sample size being 15 in statistical analysis?

A sample size of 15 is relatively small, which means the t-distribution should be used instead of the normal distribution when making inferences about the population mean.

How does the sample mean relate to the confidence interval in this context?

The sample mean serves as the point estimate for the population mean, and along with the sample standard deviation and sample size, it is used to construct a confidence interval to estimate the population mean.

What assumptions are necessary when analyzing a sample of size 15 from a normally distributed population?

Key assumptions include that the data are randomly sampled, the population is normally distributed, and the observations are independent of each other.

How does the sample mean help in detecting deviations from normality in the population?

While the sample mean itself doesn't directly detect deviations from normality, examining its distribution over multiple samples can reveal whether the population is approximately normal; with a small sample size, normality assumption is especially important.

What is the importance of knowing the sample mean in the context of statistical inference?

Knowing the sample mean allows statisticians to estimate the population mean, test hypotheses, and construct confidence intervals, forming the foundation of many inferential statistical procedures.

Additional Resources

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Introduction

In statistical inference, understanding the properties of sample means drawn from a population is fundamental. When a random sample of size 15 is taken from a normally distributed population, and the sample reveals a specific mean, several critical questions arise. How reliable is this sample mean? What can it tell us about the population parameter? And what statistical methods should be employed to interpret such data accurately? This article delves into these questions, examining the significance of sample size, population distribution assumptions, and the inferential techniques that allow statisticians to draw meaningful conclusions.

Context and Importance of Sampling from a Normal Population

Sampling forms the bedrock of inferential statistics, enabling researchers to make educated guesses about a population based on a subset. When the population is known or assumed to be normally distributed, this simplifies many statistical procedures, especially hypothesis testing and confidence interval estimation.

In this context, taking a sample of size 15 can be viewed as a manageable yet modest dataset that provides insights into the population mean. The sample mean, denoted as $\bar{x}$, acts as a point estimate for the population mean μ . The fundamental question is: given the sample mean, what can we infer about μ ?

The Significance of Sample Size ($n=15$)

Sample size significantly influences the reliability and precision of statistical estimates. For small samples (typically $(n < 30)$), the variability of the sample mean tends to be higher, and the distribution of the sample mean follows a Student's t-distribution rather than a normal distribution, especially if the population standard deviation (σ) is unknown.

Key considerations for $(n=15)$:

- Sampling Variability: The standard error of the mean (SEM), calculated as $(\mathrm{SEM} = \frac{\sigma}{\sqrt{n}})$, diminishes as (n) increases. With $(n=15)$, the SEM is relatively larger compared to larger samples, indicating higher variability.
- Distributional Assumption: For small samples, assuming normality of the population is critical. If the population is indeed normal, the t-distribution applies; if not, the results may be skewed.
- Degrees of Freedom: The t-distribution in this case has $(n-1=14)$ degrees of freedom, influencing the critical values used in hypothesis testing or confidence intervals.

Investigating the Sample Mean: Hypotheses and Confidence Intervals

When a sample mean is observed, statisticians often formulate hypotheses about the population mean, such as testing whether (μ) equals a specific value. This involves:

- Null Hypothesis (H_0) : $(\mu = \mu_0)$
- Alternative Hypothesis (H_1) : $(\mu \neq \mu_0)$ (two-tailed), or one-sided variants

Constructing Confidence Intervals (CI):

A common approach is to compute a confidence interval around the sample mean, providing a range in which the true population mean likely resides with a specified confidence level (e.g., 95%).

$$\text{CI} = \bar{x} \pm t_{(1-\alpha/2, n-1)} \times \frac{s}{\sqrt{n}}$$

Where:

- s is the sample standard deviation
- $t_{(1-\alpha/2, n-1)}$ is the t-critical value for the desired confidence level

Implications:

- Larger intervals suggest more uncertainty, typical with smaller samples.
- The precision of the estimate depends heavily on the variability in the data and the sample size.

Statistical Methods for Small Sample Analysis

Given the sample size of 15, the primary statistical tools include:

1. Student's t-Distribution

This distribution accounts for the extra variability introduced by small samples and unknown population variance. It provides critical values for hypothesis testing and confidence interval calculations.

2. Estimation of Population Mean (μ)

- Using the sample mean $(\bar{x})$ as an unbiased point estimator.
- Constructing a $(100(1-\alpha)\%)$ confidence interval to quantify the uncertainty.

3. Hypothesis Testing

For example, testing whether the population mean exceeds a certain threshold involves calculating the t-statistic:

$$t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}}$$

and comparing it with critical t-values.

Challenges and Caveats

While sampling from a normal population simplifies analysis, several challenges persist:

- Assumption of Normality: The validity of the t-distribution hinges on the population's normality, especially for small (n) . Deviations can lead to inaccurate inferences.
- Sample Variability: Small samples are more susceptible to outliers, which can skew results.
- Estimating (σ) : With only one sample, the estimate of the population standard deviation (s) may be imprecise.

Practical Applications and Case Studies

Several fields benefit from such analyses:

- Manufacturing Quality Control: Small batch testing for product consistency.
- Medical Trials: Early-phase studies with limited subjects.
- Environmental Monitoring: Small-scale sampling of pollutants.

In each case, understanding the sample mean's implications helps inform decision-making.

Interpreting the Sample Mean in Context

Suppose the sample mean is notably higher or lower than the expected population mean. In that case, the researcher must determine whether this difference is statistically significant or likely due to sampling variability.

- Significant Result: When the confidence interval does not contain μ_0 , or the hypothesis test rejects H_0 .
- Inconclusive Result: When the interval includes μ_0 , indicating insufficient evidence to draw firm conclusions.

Conclusion

A random sample of size 15 from a normally distributed population offers valuable insights but also demands careful statistical handling. The sample mean acts as a pivotal point estimate, but its reliability depends on factors such as variability, the assumption of normality, and the use of appropriate inferential methods like the t-distribution.

In practice, small-sample analysis necessitates cautious interpretation, acknowledging the increased uncertainty inherent to limited data. Nonetheless, with rigorous application of statistical techniques—confidence intervals, hypothesis testing, and proper acknowledgment of assumptions—researchers can derive meaningful conclusions about the underlying population.

This exploration underscores the importance of sample size considerations, distributional assumptions, and methodological rigor in statistical inference, particularly when working with limited data from

normally distributed populations.

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