A Rectangular Box Without A Lid Is To Be Made From 12 M Ofcardboard. Find The Maximum Volume Of Such

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Introduction

Creating a box with maximum volume using a limited amount of material is a classic optimization problem in calculus and geometry. In this particular problem, we are tasked with designing a rectangular box without a lid, using only 12 meters of cardboard. The goal is to determine the dimensions that will yield the maximum possible volume. This scenario involves understanding the relationship between surface area and volume, setting up an appropriate mathematical model, and applying calculus techniques such as differentiation to find the optimal dimensions.

Problem Restatement and Assumptions

Before delving into calculations, let's clarify the problem's parameters and assumptions:

- The box is rectangular and has no lid.
- The total cardboard used (surface area) is 12 meters.
- The box is open at the top.
- The dimensions are length (l), width (w), and height (h), all positive real numbers.
- The goal is to maximize the volume (V) of the box.

Given these assumptions, our task is to find the dimensions (l, w, h) that maximize the volume under the given surface area constraint.

Formulating the Mathematical Model

Surface Area Constraint

Since the box has no lid, the surface area (SA) comprises:

- The bottom: $l \times w$
- The four vertical sides: $2(l \times h) + 2(w \times h)$

Total surface area:

$$
SA = l w + 2 l h + 2 w h
$$

Given:

$$
SA = 12 \text{ meters}
$$

So,

$$
l w + 2 l h + 2 w h = 12
$$

Volume Expression

The volume (V) of the box is:

$$
V = l \times w \times h
$$

Our goal is to maximize V subject to the surface area constraint.

Reducing the Problem

To optimize V, it's convenient to express it in terms of fewer variables. Since the surface area constraint involves l, w, and h, we can reduce the problem by expressing one variable in terms of the others.

A practical approach is to fix the relationship between l and w or to assume symmetry. For simplicity, assume the base is square, i.e.,

$$
l = w
$$

This is a common assumption in such problems to reduce complexity, and often the maximum volume occurs when the base is square.

Thus:

$$
l = w = x
$$

Now, the surface area becomes:

$$
SA = x \times x + 2 x h + 2 x h = x^2 + 4 x h
$$

and the volume:

$$
V = x \times x \times h = x^2 h
$$

Given that:

$$
x^2 + 4 x h = 12
$$

we can solve for h:

$$
4 x h = 12 - x^2
$$

$$
h = \frac{12 - x^2}{4 x}
$$

The volume function V in terms of x:

$$
V(x) = x^2 \times h = x^2 \times \frac{12 - x^2}{4 x} = \frac{x^2 (12 - x^2)}{4 x}
$$

Simplify:

$$
V(x) = \frac{x (12 - x^2)}{4}
$$

$$
V(x) = \frac{12 x - x^3}{4}
$$

Now, the domain of x is constrained by the requirement that h > 0:

$$
h > 0 \Rightarrow 12 - x^2 > 0 \Rightarrow x^2 < 12 \Rightarrow x < \sqrt{12} \approx 3.464
$$

and since x > 0, the domain is:

\[

$$
0 < x < \sqrt{12}
$$

Optimizing the Volume

Step 1: Derivative of V(x)

Differentiate V(x) with respect to x:

$$
V'(x) = \frac{1}{4} (12 - 3 x^2)
$$

Step 2: Find Critical Points

Set the derivative to zero to find potential maxima:

$$
V'(x) = 0 \Rightarrow 12 - 3 x^2 = 0
$$

$$
3 x^2 = 12
$$

$$
x^2 = 4
$$

$$
x = 2 \quad (\text{since } x > 0)
$$

Step 3: Verify the Critical Point

Calculate the volume at $x=2$:

$$
h = \frac{12 - (2)^2}{4 \times 2} = \frac{12 - 4}{8} = \frac{8}{8} = 1
$$

Volume:

$$
V(2) = \frac{12 \times 2 - (2)^3}{4} = \frac{24 - 8}{4} = \frac{16}{4} = 4
$$

Step 4: Check the Endpoints

- As $x \to 0^+$, volume $V \to 0$.
- As $x \to \sqrt{12}^-$, $h \to 0^+$, volume approaches zero again.

Therefore, the critical point at $x=2$ gives the maximum volume.

Final Dimensions and Maximum Volume

- Base sides:

$$
l = w = 2 \text{ meters}
$$

- Height:

$$
h = 1 \text{ meter}
$$

Maximum volume:

$$
V_{max} = 4 \text{ cubic meters}
$$

Conclusion

By assuming a square base, we've determined that the maximum volume of a lidless rectangular box constructed from 12 meters of cardboard occurs when the base sides are 2 meters each, and the height is 1 meter. The maximum volume achievable under these conditions is 4 cubic meters.

This problem illustrates the importance of setting up an appropriate mathematical model, simplifying assumptions to reduce complexity, and applying calculus techniques such as differentiation to solve optimization problems. Such approaches are fundamental in design, engineering, and resource management where maximizing efficiency within constraints is essential.

Extensions and Considerations

- If the problem does not assume a square base, the optimization becomes more complex and involves multiple variables.
- Real-world applications may include material thickness, structural stability, or additional constraints, which would modify the model.
- The method demonstrated here can be adapted to similar problems involving surface area and volume optimization under different conditions.

Frequently Asked Questions

What is the problem asking for in the context of creating a box from 12 meters of cardboard?

The problem asks to determine the maximum possible volume of a rectangular box without a lid that can be made using 12 meters of cardboard.

What are the variables involved in maximizing the volume of the box?

The variables typically include the length (l), width (w), and height (h) of the box, with the constraint that the total surface area used equals 12 meters of cardboard.

How is the surface area constraint expressed mathematically for this box?

Since the box has no lid, the surface area consists of the bottom and four sides: $l \times w + 2(l+h) + 2(w+h) = 12$ meters.

What mathematical method is suitable for finding the maximum volume under the given constraint?

Using calculus, specifically setting up a volume function and applying optimization techniques with constraint handling such as Lagrange multipliers or substitution, is suitable.

What is the formula for the volume of the box, and how does it relate to the dimensions?

The volume $V = l \times w \times h$, where l, w, and h are the length, width, and height of the box, respectively, and our goal is to maximize V given the surface area constraint.

What is the key insight for maximizing the volume with a fixed surface area in this problem?

The key insight is that for a given surface area, the dimensions that maximize volume often involve equal or proportionally optimal dimensions, which can be found by setting derivatives to zero and solving for the variables.

Additional Resources

A Rectangular Box Without A Lid Is To Be Made From 12 M Of Cardboard: An In-Depth Analysis

Designing a box with maximum volume using a limited amount of material is a classic problem in optimization and geometry. Specifically, creating a rectangular box without a lid from 12 meters of cardboard involves strategic planning to ensure that the volume enclosed is as large as possible, given the constraint on the surface area. This problem not only has practical implications in packaging and manufacturing but also offers a fascinating insight into the principles of calculus, optimization, and geometric reasoning. In this article, we will explore the problem in detail, breaking down the mathematical formulation, solving the optimization problem step by step, and discussing the features, advantages, and potential limitations of the solution.

Understanding the Problem

Before diving into calculations and mathematical modeling, it’s important to clearly understand what the problem entails.

- Objective: Maximize the volume of a rectangular box without a lid.
- Material Constraints: The box is to be made from 12 meters of cardboard.
- Design Specifications:
- The box is rectangular (length, width, height).
- The box has no lid (top open).
- Given Data:
- Total surface area of cardboard used: 12 square meters.

The key challenge here is to determine the dimensions—length (l), width (w), and height (h)—that will produce the maximum volume within the surface area constraint.

Mathematical Modeling of the Problem

To solve this problem, we need to formulate it mathematically.

Variables Definition

Let:

- l = length of the box
- w = width of the box
- h = height of the box

Surface Area Constraint

Since the box has no lid, the surface area comprises:

- The bottom: $l \times w$
- The four vertical sides: $2lh + 2wh$

Total surface area (SA):

$$
SA = lw + 2lh + 2wh
$$

Given:

$$
lw + 2lh + 2wh = 12

\]

Volume Expression

The volume \(V \) of the box:

\[

V = l \times w \times h

\]

Our goal:

\[

\text{Maximize } V = lwh

\]

subject to:

\[

lw + 2lh + 2wh = 12

\]

Optimization Strategy

The problem involves three variables, but typically, such problems can be simplified by expressing the volume in terms of two variables and then optimizing with respect to the third.

Expressing l in terms of l and w

From the surface area constraint:

$$
lw + 2lh + 2wh = 12
$$

Rearranged:

$$
2lh + 2wh = 12 - lw
$$

Factor:

$$
2h(l + w) = 12 - lw
$$

Solve for h:

$$
h = \frac{12 - lw}{2(l + w)}
$$

Now, the volume becomes:

$$
V = l \times w \times h = l w \times \frac{12 - lw}{2(l + w)}
$$

This expression involves two variables, l and w. To further simplify, we can utilize symmetry or set specific relationships between l and w.

Assuming Symmetry: Setting $l = w$

In many optimization problems involving rectangular shapes, symmetry often leads to the optimal solution. Let's assume:

$$
l = w = x
$$

Thus:

$$
h = \frac{12 - x^2}{4x}
$$

The volume becomes:

$$
V = x \times x \times h = x^2 \times \frac{12 - x^2}{4x} = \frac{x(12 - x^2)}{4}
$$

Simplify:

$$
V(x) = \frac{12x - x^3}{4}
$$

Maximizing the Volume Function

Now, $V(x)$ is a single-variable function:

$$

V(x) = \frac{12x - x^3}{4}

\]

To find the maximum volume, take the derivative with respect to \(x \):

\[

V'(x) = \frac{12 - 3x^2}{4}

\]

Set derivative to zero:

\[

\frac{12 - 3x^2}{4} = 0

\]

\[

12 - 3x^2 = 0

\]

\[

3x^2 = 12

\]

\[

x^2 = 4

\]

\[

x = 2 \quad (\text{considering positive dimensions})

\]

Verify that this critical point corresponds to a maximum by checking the second derivative:

\[

V''(x) = \frac{-6x}{4} = -\frac{3x}{2}

\]

At \(x=2 \):

\[

$$
V''(2) = -\frac{3 \times 2}{2} = -3 < 0
$$

Since second derivative is negative, $V(x)$ has a maximum at $x=2$.

Determining the Dimensions and Maximum Volume

Recall:

$$
l = w = x = 2 \text{ meters}
$$

and

$$
h = \frac{12 - x^2}{4x} = \frac{12 - 4}{8} = \frac{8}{8} = 1 \text{ meter}
$$

Dimensions of the box:

- Length $l = 2$ meters
- Width $w = 2$ meters
- Height $h = 1$ meter

Maximum Volume:

$$
V_{max} = l \times w \times h = 2 \times 2 \times 1 = 4 \text{ cubic meters}
$$

Features, Pros, and Cons of the Solution

Features:

- The solution efficiently maximizes the volume with a straightforward symmetrical assumption.
- It leverages calculus to find critical points, ensuring an optimal solution.
- The problem demonstrates practical applications of geometric optimization.

Pros:

- Optimal Use of Material: Achieves maximum volume with limited cardboard.
- Simplicity: Assumption $l = w$ simplifies calculations without loss of generality.
- Educational Value: Illustrates the application of derivatives in real-world problems.

Cons:

- Assumption of Symmetry: The solution assumes $l = w$; in some cases, asymmetric dimensions might yield better results if the problem constraints differ.
- Limited to Specific Constraints: The model assumes a fixed surface area; changes in constraints require re-derivation.
- Neglects Manufacturing Tolerances: Practical considerations such as material thickness or cutting imperfections are not considered.

Alternative Approaches and Further Considerations

While the symmetric assumption provides a neat solution, exploring more general cases where $l \neq w$ might yield slightly different dimensions, especially if the dimensions are constrained by other factors. For instance, setting $l \neq w$ and applying Lagrange multipliers could provide a more general solution.

Furthermore, considering the problem in three variables without symmetry could involve setting partial derivatives and solving for critical points, though the symmetry assumption simplifies this process.

Conclusion

The problem of maximizing the volume of a rectangular box without a lid from 12 meters of cardboard is a classic example of geometric optimization. Through mathematical modeling, assumption of symmetry, and derivative analysis, the optimal dimensions are determined to be 2 meters by 2 meters for the base and 1 meter in height, resulting in a maximum volume of 4 cubic meters.

This problem showcases the practical application of calculus in design and manufacturing, emphasizing the importance of mathematical reasoning in optimizing real-world objects. While the solution hinges on specific assumptions, it provides valuable insights into how constraints shape optimal designs and how mathematical tools can be employed to find the best possible outcomes within given limitations.

In summary:
- The maximum volume achievable is 4 cubic meters.
- The optimal dimensions are length = 2 meters, width = 2 meters, height = 1 meter.
- The problem illustrates the power of calculus and geometric reasoning in solving real-world optimization challenges.

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