(a) Use Three Iterations Of The Bisection Method To Find An Approximate Solution For X Cos X 2x + 3x

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Introduction to the Bisection Method

The bisection method is a straightforward and reliable numerical technique used to find roots of continuous functions. It is especially useful when the function changes sign over an interval, indicating the presence of at least one root within that range. In this article, we will explore how to apply three iterations of the bisection method to approximate the solution for the function:

$$ f(x) = x \cos x - 2x + 3x $$

which simplifies to:

$$ f(x) = x \cos x + x $$

before proceeding to detailed calculations.

Understanding the Function and Its Roots

Analyzing the Function $f(x)$

Let's examine the function:

$$
f(x) = x \cos x + x
$$

which can be factored as:

$$
f(x) = x (\cos x + 1)
$$

This insight simplifies our analysis for roots:

- Roots occur when $f(x) = 0$, i.e., when $x (\cos x + 1) = 0$.

- Therefore, the roots are at:

1. $x = 0$
2. $\cos x + 1 = 0 \Rightarrow \cos x = -1$

Since $\cos x = -1$ at $x = (2n + 1)\pi$, where n is an integer, roots are at:

$$
x = \pm \pi, \pm 3\pi, \pm 5\pi, \dots
$$

Given $f(0) = 0$, the root at zero is trivial. To demonstrate the bisection method, we focus on a specific interval where $f(x)$ crosses zero, for example, between $x = 1$ and $x = 2$.

Selecting the Interval for Bisection

Choosing the Interval $[a, b]$

To apply the bisection method, we need an interval where $f(a)$ and $f(b)$ have opposite signs:

- Calculate $f(1)$:

$$
f(1) = 1 \times \cos 1 + 1 \approx 1 \times 0.5403 + 1 \approx 1.5403
$$

- Calculate $f(2)$:

$$
f(2) = 2 \times \cos 2 + 2 \approx 2 \times (-0.4161) + 2 \approx -0.8322 + 2 = 1.1678
$$

Both are positive, so no root between 1 and 2.

Next, try $[1, 3]$:

- $f(3)$:

$$
f(3) = 3 \times \cos 3 + 3 \approx 3 \times (-0.9899) + 3 \approx -2.9697 + 3 = 0.0303
$$

- $f(2)$:

$$
f(2) \approx 1.1678
$$

Both positive again, so no sign change between 2 and 3.

Try $[2, 4]$:

- $f(4)$:

$$
f(4) = 4 \times \cos 4 + 4 \approx 4 \times (-0.6536) + 4 \approx -2.6144 + 4 = 1.3856
$$

- $f(3)$:

$$
0.0303
$$

Both positive; no sign change.

Try $[3, 4]$:

- $f(3) \approx 0.0303$
- $f(4) \approx 1.3856$

Still both positive. Let's check $[3, 5]$:

- $f(5)$:

$$
f(5) = 5 \times \cos 5 + 5 \approx 5 \times 0.2837 + 5 \approx 1.4185 + 5 = 6.4185
$$

No sign change; so the root is not between 3 and 5.

Now, check $[4, 5]$:

- $f(4) \approx 1.3856$
- $f(5) \approx 6.4185$

No sign change.

Alternatively, check $[-1, 0]$:

- $f(-1)$:

$$
-1 \times \cos(-1) + (-1) = -1 \times 0.5403 - 1 \approx -0.5403 - 1 = -1.5403
$$

- $f(0)$:

$$

$$
0 \times \cos 0 + 0 = 0
$$

Since $f(-1) < 0$ and $f(0) = 0$, the root at zero is within $[-1, 0]$.

Given the simplicity, we can focus on the root at $x=0$, which is exact. For demonstration, suppose we want to find an approximate root between $x=0.5$ and $x=1$:

- $f(0.5)$:

$$
0.5 \times \cos 0.5 + 0.5 \approx 0.5 \times 0.8776 + 0.5 \approx 0.4388 + 0.5 = 0.9388
$$

- $f(1)$:

$$
1 \times 0.5403 + 1 \approx 1.5403
$$

Both positive; no sign change.

Similarly, between -1 and 0:

- $f(-0.5)$:

$$
-0.5 \times \cos(-0.5) + (-0.5) \approx -0.5 \times 0.8776 - 0.5 = -0.4388 - 0.5 = -0.9388
$$

- $f(0)$:

$$
0
$$

Since $f(0) = 0$, the root at zero is confirmed.

Conclusion: For the purpose of this article, let's consider the root near $x \approx 1.5$ where the function crosses zero. Let's check $f(1.5)$:

$$
f(1.5) = 1.5 \times \cos 1.5 + 1.5 \approx 1.5 \times 0.0707 + 1.5 \approx 0.1061 + 1.5 = 1.6061
$$

And $f(2)$:

$$
f(2) \approx 1.1678
$$

Both positive. Let's check $f(2.5)$:

$$
2.5 \times \cos 2.5 + 2.5 \approx 2.5 \times -0.8011 + 2.5 \approx -2.00275 + 2.5 = 0.49725
$$

Still positive, but decreasing.

Check $f(3)$:

$$
\approx 0.0303
$$

Close to zero. Now, $f(3.5)$:

$$
3.5 \times \cos 3.5 + 3.5 \approx 3.5 \times -0.9365 + 3.5 \approx -3.2778 + 3.5 = 0.2222
$$

Positive. Since the function remains positive in these ranges, for the purpose of applying the bisection method, let's pick the interval $[3, 4]$:

- $f(3) \approx 0.0303$
- $f(4) \approx 1.3856$

Both positive, so no root here.

Alternatively, consider the root at $\pi \approx 3.1416$:

- $f(\pi) = \pi \times \cos \pi + \pi = \pi \times -1 + \pi = -\pi + \pi = 0$

This confirms the root at $x=\pi$.

Given this, to demonstrate the bisection method, we can choose the interval $[3, 4]$:

- Since $f(3) \approx 0.0303 > 0$
- $f(4) \approx 1.3856 > 0$

No sign change, but at $x=3.14$:

$$
f(3.14) \approx 3.14 \times \cos 3.14 + 3.14 \approx 3.14 \times -1 + 3.14 = -3.14 + 3.14 = 0
$$

which is exact at π. For the purpose of this article, let's proceed with the root at $x=\pi$, approximately 3.14, and demonstrate three iterations to refine the approximation.

Frequently Asked Questions

What is the goal of applying the Bisection Method to the equation X Cos X - 2X + 3X = 0?

The goal is to find an approximate root of the equation by iteratively narrowing down an interval where the function changes sign, leading us closer to the solution.

How do you set up the initial interval for applying the Bisection Method to this equation?

You choose two points, say a and b, such that the function values at these points have opposite signs, ensuring a root lies between them.

What is the first step in the Bisection Method after selecting initial interval endpoints?

Calculate the midpoint of the interval, evaluate the function at this point, and determine which subinterval contains the root based on the sign change.

Can you demonstrate the first iteration for the equation X Cos X - 2X + 3X using initial interval [0, 1]?

Yes. First, evaluate at endpoints: f(0)=0, and f(1)=1Cos1 - 21 + 31. Calculate f(1). Then, find the midpoint x=0.5, evaluate f(0.5), and determine which subinterval contains the root based on the sign.

How do the second and third iterations improve the approximation of the root?

In each iteration, the interval is halved, and the midpoint is evaluated. The process narrows down the interval containing the root, improving the approximation with each step.

What are the advantages of using the Bisection Method for this type of equation?

It is simple, reliable, and guarantees convergence if the initial interval brackets a root, making it suitable for equations like X Cos X - 2X + 3X.

Are there any limitations to applying three iterations of the Bisection Method in this context?

Yes, three iterations may not provide a very precise solution, especially if the initial interval is large; more iterations are often needed for higher accuracy.

How do you determine when to stop iterating in the Bisection Method?

You stop when the interval width is sufficiently small or when the function value at the midpoint is close enough to zero, based on the desired precision.

What is the approximate solution for X after three Bisection iterations for this equation?

After three iterations, the approximate solution is the midpoint of the current interval, which is narrowed down through the process—its exact value depends on the specific calculations at each step.

Can the Bisection Method be combined with other techniques to improve accuracy for this problem?

Yes, after a few Bisection iterations, methods like Newton-Raphson can be applied to refine the root estimate further, combining reliability with faster convergence.

Additional Resources

Use Three Iterations Of The Bisection Method To Find An Approximate Solution For X Cos X 2x + 3x

When tackling nonlinear equations in mathematics and engineering, the bisection method stands out as a reliable and straightforward numerical technique. In this guide, we will explore how to use three iterations of the bisection method to find an approximate solution for the equation \(x \cos x + 2x + 3x = 0 \). This problem exemplifies the practical utility of iterative root-finding methods, particularly when analytical solutions are challenging or impossible to derive.

Understanding the Bisection Method

The bisection method is a bracketing root-finding technique that repeatedly halves an interval to narrow down the location of a root in a continuous function. Its simplicity, guaranteed convergence (for continuous functions where the function changes sign over the interval), makes it a popular choice in numerical analysis.

Why Use the Bisection Method?

- Reliability: It guarantees convergence provided the initial interval brackets a root.

- Simplicity: Easy to implement and understand.
- Robustness: Works well with functions that are continuous and have known sign changes over an interval.

Analyzing the Equation

The function we are considering is:

$$f(x) = x \cos x + 2x + 3x$$

Simplify the expression:

$$f(x) = x \cos x + 5x$$

Our goal is to find an approximate value of x such that $f(x) = 0$.

Step 1: Choosing an Initial Interval

To apply the bisection method, we first need to identify an interval $[a, b]$ where the function changes sign—that is, $f(a) \times f(b) < 0$.

Selecting the Interval

Let's evaluate $f(x)$ at some points:

- At $x=0$:

$$ f(0) = 0 \times \cos 0 + 5 \times 0 = 0 + 0 = 0 $$

Interesting—here, $f(0) = 0$, so 0 is a root. But since the goal is to perform iterations, we can choose an interval around zero, say $[-1, 1]$.

- At $x=1$:

$$ f(1) = 1 \times \cos 1 + 5 \times 1 \approx 1 \times 0.5403 + 5 = 0.5403 + 5 = 5.5403 $$

- At $x=-1$:

$$ f(-1) = -1 \times \cos (-1) + 5 \times (-1) = -1 \times 0.5403 - 5 = -0.5403 - 5 = -5.5403 $$

Since $f(-1) < 0$ and $f(1) > 0$, the function changes sign over $[-1, 1]$. Therefore, the initial interval is $[-1, 1]$.

Step 2: Performing the First Iteration

1. Calculate the midpoint:

$$ c_1 = \frac{a + b}{2} = \frac{-1 + 1}{2} = 0 $$

2. Evaluate $f(c_1)$:

$$ f(0) = 0 \times \cos 0 + 5 \times 0 = 0 $$

Since $f(0) = 0$, we've found an exact root at $x=0$. For educational purposes, let's pretend $f(0) \neq 0$ (say, initial interval $[-1, 2]$) and proceed to illustrate the process.

Suppose initial interval: $[-1, 2]$.

- At $x=2$:

$$ f(2) = 2 \times \cos 2 + 5 \times 2 \approx 2 \times (-0.4161) + 10 = -0.8322 + 10 = 9.1678 $$

- At $x=-1$:

$$ f(-1) \approx -0.5403 - 5 = -5.5403 $$

Since $f(-1) < 0$ and $f(2) > 0$, the root lies within $[-1, 2]$.

3. First midpoint:

$$ c_1 = \frac{-1 + 2}{2} = 0.5 $$

- Evaluate $f(0.5)$:

$$ f(0.5) = 0.5 \times \cos 0.5 + 5 \times 0.5 \approx 0.5 \times 0.8776 + 2.5 = 0.4388 + 2.5 = 2.9388 $$

Since $f(-1) < 0$ and $f(0.5) > 0$, the root is between -1 and 0.5.

Step 3: Performing the Second Iteration

- Update interval: $[-1, 0.5]$

- Calculate midpoint:

$$ c_2 = \frac{-1 + 0.5}{2} = -0.25 $$

- Evaluate $f(-0.25)$:

$$ f(-0.25) = -0.25 \times \cos(-0.25) + 5 \times (-0.25) $$

$$ \cos(-0.25) = \cos 0.25 \approx 0.9689 $$

$$ f(-0.25) = -0.25 \times 0.9689 - 1.25 = -0.2422 - 1.25 = -1.4922 $$

- Since $f(-0.25) < 0$ and $f(0.5) > 0$, the root lies within $[-0.25, 0.5]$.

Step 4: Performing the Third Iteration

- Update interval: $[-0.25, 0.5]$

- Calculate midpoint:

$$ c_3 = \frac{-0.25 + 0.5}{2} = 0.125 $$

- Evaluate $f(0.125)$:

$$ f(0.125) = 0.125 \times \cos 0.125 + 5 \times 0.125 $$

$$ \cos 0.125 \approx 0.9922 $$

$$ f(0.125) = 0.125 \times 0.9922 + 0.625 \approx 0.1240 + 0.625 = 0.7490 $$

- Since $f(-0.25) < 0$ and $f(0.125) > 0$, the root is within $[-0.25, 0.125]$.

Summary of the Three Iterations

Iteration	Interval $[a, b]$	Midpoint c	$f(c)$	Sign of $f(c)$	Next Interval
1	$[-1, 2]$	0.5	2.9388	Positive	$[-1, 0.5]$
2	$[-1, 0.5]$	-0.25	-1.4922	Negative	$[-0.25,$

0.5]\) |
| 3 | $[-0.25, 0.5]$ | 0.125 | 0.7490 | Positive | $[-0.25, 0.125]$ |

After three iterations, the root is approximately within $[-0.25, 0.125]$. The approximate root could be taken as the midpoint of this interval:

$$ x \approx \frac{-0.25 + 0.125}{2} = -0.0625 $$

Practical Tips for Applying the Bisection Method

1. Initial Interval Selection: Always verify that the function changes sign over the interval. Use graphing tools or evaluation at multiple points to identify suitable brackets.

2. Iteration Count: The number of iterations needed depends on the desired accuracy. Each iteration halves the interval, so after n iterations, the error is roughly $\frac{b - a}{2^n}$.

3. Stopping Criteria: Besides fixed iterations, consider stopping when the interval width is below a certain tolerance or when $f(c)$ is sufficiently close to zero.

4. Function Behavior: Ensure the function is continuous on the interval to guarantee convergence.

Final Remarks

Using three iterations of the bisection method provides a significantly refined approximation of the root of the equation $x \cos x + 2x + 3x = 0$. While the initial estimate gave a broad interval, iterative halving narrows down the possible root region efficiently and reliably. This approach exemplifies how simple numerical methods

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