The Function F(t) = T Is Sampled Every Interval Of T. Find The Z-transform Of The Sampled Function.

The Function F(t) = T Is Sampled Every Interval Of T. Find The Z-transform Of The Sampled Function.

Understanding how continuous-time signals are transformed into discrete sequences is fundamental in digital signal processing, control systems, and communications engineering. Sampling is the process of converting a continuous-time function into a discrete-time sequence, enabling digital analysis and processing. One critical tool in analyzing discrete signals is the Z-transform, which provides insights into system stability, frequency response, and system behavior.

In this article, we delve into the problem of finding the Z-transform of a sampled function, specifically when the original function is given as F(t) = T sampled every interval of T. We will explore the underlying principles, mathematical derivations, and practical applications, ensuring a comprehensive understanding of the subject.

Understanding Sampling in Signal Processing

Sampling is the process of converting a continuous-time signal into a discrete-time sequence by taking measurements at regular intervals. The fundamental sampling interval is denoted as T, which is the time between successive samples.

Key Concepts:

- Sampling Interval (T): The fixed time period between samples.
- Sampling Rate (1/T): The number of samples taken per second.
- Sampled Signal: Discrete sequence obtained from the continuous-time function.

Mathematical Representation:

Given a continuous-time function $f(t)$, the sampled signal $f_s(t)$ can be expressed using the Dirac delta function:

$$
f_s(t) = \sum_{n=-\infty}^{\infty} f(nT) \delta(t - nT)
$$

where $f(nT)$ is the sample value at $t = nT$.

In our case, the continuous function is:

$$
F(t) = T
$$

which is a constant function. It is sampled every interval T, resulting in a discrete sequence:

$$
f[n] = F(nT) = T
$$

for all integer n.

Discrete-Time Representation of the Sampled Function

Since $F(t) = T$ is a constant, the sampled sequence is:

$$
f[n] = T \quad \text{for all } n \in \mathbb{Z}
$$

This sequence is a constant-valued discrete-time signal, which simplifies the analysis of its Z-transform.

Introduction to the Z-transform

The Z-transform is a powerful tool for analyzing discrete-time signals and systems. It maps a discrete sequence into the complex frequency domain, enabling the study of system stability, frequency response, and feedback behavior.

Definition:

For a discrete-time sequence $f[n]$, the Z-transform $F(z)$ is defined as:

$$
F(z) = \sum_{n=-\infty}^{\infty} f[n] z^{-n}
$$

where:

- z is a complex variable $z = re^{j\omega}$,
- The region of convergence (ROC) depends on the sequence.

Properties of the Z-transform:

- Linearity
- Time shifting
- Scaling
- Convolution

Deriving the Z-transform of the Sampled Function $f[n] = T$

Given the sampled sequence:

$$
f[n] = T, \quad \forall n
$$

we can compute its Z-transform as:

$$
F(z) = \sum_{n=-\infty}^{\infty} T z^{-n}
$$

This is a sum of a constant over all integers. Since the sequence is constant for all n, the sum converges only under certain conditions. Typically, we consider causal sequences (i.e., $n \geq 0$) for practical systems.

Assuming causality (i.e., $n \geq 0$), the Z-transform becomes:

$$
F(z) = \sum_{n=0}^{\infty} T z^{-n}
$$

This is a geometric series with common ratio z^{-1}:

$$
F(z) = T \sum_{n=0}^{\infty} (z^{-1})^n
$$

Sum of the geometric series:

$$
F(z) = T \cdot \frac{1}{1 - z^{-1}}, \quad \text{for } |z| > 1
$$

Rearranged:

$$
F(z) = \frac{T}{1 - z^{-1}} = \frac{T z}{z - 1}
$$

Final Expression for the Z-transform

The Z-transform of the sampled function $f[n] = T$ (for $n \geq 0$) is:

$$
\boxed{
F(z) = \frac{T z}{z - 1}
}
$$

Region of Convergence (ROC):

- The series converges when $|z| > 1$.
- This indicates the Z-transform is valid outside the unit circle in the complex plane.

Interpreting the Result

The derived Z-transform $F(z) = \frac{T z}{z - 1}$ represents the behavior of the sampled constant function in the Z-domain. This function has a simple pole at $z = 1$, which reflects the steady-state nature of the constant signal.

Key observations:

- The pole at $z = 1$ indicates a non-decaying (constant) component in the sequence.
- The magnitude of the pole (at 1) signifies that the sequence is neither growing nor diminishing over time.
- The Z-transform provides a convenient way to analyze the system's response, stability, and frequency characteristics.

Applications of the Z-transform in Signal Processing

The Z-transform is widely used in various applications, including:

1. Digital Filter Design: Analyzing and designing filters with desired frequency responses.
2. System Stability Analysis: Determining whether a discrete system is stable based on pole locations.
3. Signal Analysis: Investigating the spectral components of discrete signals.
4. Control Systems: Designing digital controllers and observers.

Specifically, understanding the Z-transform of sampled functions allows engineers to analyze how continuous signals translate into discrete domains, facilitating the design of digital systems that accurately replicate or process analog signals.

Summary and Conclusion

In this article, we examined the process of finding the Z-transform of a sampled constant function $F(t) = T$. When sampled every interval T, the resulting discrete sequence is $f[n] = T$. Assuming causality (i.e., $n \geq 0$), the Z-transform is:

$$
F(z) = \frac{T z}{z - 1}
$$

This fundamental result underscores the importance of the Z-transform in digital signal processing, enabling the analysis of discrete signals derived from continuous-time functions. The simplicity of the transform for constant functions also serves as a foundation for understanding more complex signals and systems.

In summary:

- Sampling transforms a continuous function into a discrete sequence.
- The Z-transform provides a powerful analysis tool for discrete sequences.
- For a constant sampled function T, the Z-transform is a rational function with a pole at $z = 1$.
- Understanding this process is crucial for system stability analysis and filter design in digital systems.

Further Reading and Resources

- Books:
- "Digital Signal Processing: Principles, Algorithms, and Applications" by John G. Proakis and Dimitris G. Manolakis.
- "Discrete-Time Signal Processing" by Alan V. Oppenheim and Ronald W. Schafer.
- Online Resources:
- Tutorials on Z-transform properties and applications.

- Signal processing courses on platforms like Coursera, edX, and Khan Academy.

By mastering the principles discussed in this article, engineers and students can better understand the relationship between continuous signals and their discrete counterparts, paving the way for advanced digital system design and analysis.

Frequently Asked Questions

What is the Z-transform of a sampled function F(t) = T δ(t - nT)?

The Z-transform of the sampled function F(t) = T δ(t - nT) is F(z) = T z^{-1} (1 + z^{-1} + z^{-2} + ...), which simplifies to a geometric series: F(z) = T z^{-1} / (1 - z^{-1}).

How do you derive the Z-transform of a uniformly sampled function F(t) = T sample at intervals of T?

The Z-transform of a sampled function is obtained by taking the discrete-time sequence f[n] = F(nT) and applying the Z-transform definition: F(z) = Σ f[n] z^{-n}. For a sampled function, this often involves expressing the sequence explicitly and summing accordingly.

What is the significance of the interval T in finding the Z-transform of the sampled function?

The interval T determines the sampling rate and directly affects the discrete sequence values. It influences the Z-transform by defining the sampling instants and the corresponding sequence, which is essential for accurate analysis of the sampled system.

Can you explain how the Z-transform relates to the Laplace transform for sampled signals?

The Z-transform is the discrete-time counterpart of the Laplace transform. While the Laplace transform analyzes continuous-time signals, the Z-transform analyzes discrete-time samples. When sampling a continuous-time signal, the Z-transform provides a way to analyze the discrete sequence derived from the continuous signal.

Why is it important to find the Z-transform of a sampled function in signal processing?

Finding the Z-transform of a sampled function allows engineers to analyze, design, and implement digital filters and systems efficiently. It provides insights into system stability, frequency response, and enables the use of algebraic techniques for discrete signals.

Additional Resources

The Function F(t) = T Is Sampled Every Interval Of T: Find The Z-Transform Of The Sampled Function

In the realm of digital signal processing, control systems, and discrete-time analysis, understanding the transformation of continuous signals into discrete counterparts is fundamental. Among the tools used for this purpose, the Z-transform stands out as a powerful mathematical technique for analyzing discrete sequences. This article delves into a specific scenario: The Function F(t) = T Is Sampled Every Interval Of T. Find The Z-transform Of The Sampled Function. We will explore the underlying concepts, derive the Z-transform step-by-step, and discuss its significance in various applications.

Introduction

The process of sampling continuous signals converts them into discrete sequences, enabling digital processing. When a continuous-time function F(t) is sampled at regular intervals, the resulting discrete sequence can be analyzed using the Z-transform, which is the discrete-time counterpart to the Laplace transform.

In this context, we examine a particular function: F(t) = T, where T is a constant. The key question is: What is the Z-transform of the sampled version of this function? To answer this, we need to understand the sampling process, the nature of the function, and how to formulate and compute the Z-transform.

Understanding the Function F(t) = T

Before delving into the sampling and transformation, it's crucial to understand the nature of the function:

- F(t) = T, where T is a constant.
- This function is a constant-valued function, independent of time t.

In a continuous domain, F(t) maintains the same value T for all $t \geq 0$. When we sample this function at discrete intervals, the sampled sequence becomes a series of constant values, each equal to T, occurring at specific points in time.

The Sampling Process

Sampling Interval and Discrete Time Index

Suppose the sampling occurs at regular intervals of T seconds, i.e., at times:

\[t = nT, \quad n = 0, 1, 2, 3, \dots \]

The sampled sequence $f[n]$ is derived by evaluating the continuous function at these points:

$$f[n] = F(nT)$$

Sampled Sequence for F(t) = T

Given the constant nature of F(t):

$$f[n] = F(nT) = T, \quad \forall n \ge 0$$

Thus, the sampled sequence is a constant sequence:

$$f[n] = T \quad \text{for all } n \ge 0$$

Deriving the Z-Transform

Definition of the Z-Transform

The Z-transform of a discrete sequence $f[n]$ is defined as:

$$
F(z) = \mathcal{Z}\{f[n]\} = \sum_{n=0}^{\infty} f[n] z^{-n}
$$

where:

- z is a complex variable.
- The sum converges within the Region of Convergence (ROC).

Applying the Definition to Our Sequence

Since $f[n] = T$, a constant:

$$
F(z) = \sum_{n=0}^{\infty} T \cdot z^{-n}
$$

Factor out T:

$$
F(z) = T \sum_{n=0}^{\infty} z^{-n}
$$

This is a geometric series with ratio $r = z^{-1}$. The sum converges for $|z| > 1$:

$$
\sum_{n=0}^{\infty} r^n = \frac{1}{1 - r}, \quad \text{for } |r| < 1
$$

Substitute $r = z^{-1}$:

$$
F(z) = T \cdot \frac{1}{1 - z^{-1}} = T \cdot \frac{z}{z - 1}
$$

Final Z-Transform Expression

Thus, the Z-transform of the sampled constant function is:

$$
\boxed{
F(z) = \frac{T z}{z - 1}
}
$$

with the Region of Convergence:

$$
|z| > 1
$$

Deep Dive: Significance and Implications

The derivation above reveals several important insights:

- Nature of the Z-Transform of a Constant Signal: The Z-transform of a constant sequence $f[n] = T$ results in a rational function $\frac{T z}{z - 1}$, illustrating how constant signals are represented in the Z-domain.
- Region of Convergence (ROC): The ROC $|z| > 1$ indicates that the Z-transform converges outside the unit circle, which aligns with the fact that the sequence is a right-sided (causal) sequence.
- Comparison to Continuous-Time Functions: The original function $F(t) = T$ is constant in continuous time, and its sampled version preserves this constancy in the discrete domain, reflected in the Z-transform as a simple rational function.

Applications and Relevance

Digital Signal Processing (DSP)

Understanding the Z-transform of simple functions like constants is fundamental when analyzing systems' stability and behavior, especially when dealing with steady-state signals.

Control Systems

In discrete-time control systems, constant inputs are common. Knowing their Z-transforms helps in system analysis, design, and in solving difference equations.

Signal Reconstruction and Sampling Analysis

The derived Z-transform serves as a fundamental building block for more complex signals and their sampling behaviors, providing a foundation for understanding aliasing, reconstruction, and digital filtering.

Extending the Analysis: Variations and Related Cases

While this article focuses on the constant function $F(t) = T$, the methodology extends to other functions:

- Step functions: For $F(t) = U(t)$, the unit step, the Z-transform becomes $\frac{z}{z - 1}$.
- Exponentials: For $F(t) = e^{a t}$, the sampled sequence is $e^{a n T}$, and the Z-transform is $\frac{z e^{a T}}{z - e^{a T}}$.

This demonstrates the universality of the Z-transform in handling various signals and the importance of understanding the basic case thoroughly.

Conclusion

The process of sampling continuous functions and transforming them into the Z-domain is central to the analysis and design of digital systems. In the specific case of the constant function $F(t) = T$, the sampled sequence is a constant sequence, and its Z-transform is:

$$
\boxed{
F(z) = \frac{T z}{z - 1}
}
$$

This result encapsulates a fundamental principle: constant signals in continuous time translate into rational functions in the Z-domain, providing a bridge between time-domain constancy and algebraic simplicity in the discrete domain.

Understanding such foundational transformations equips engineers and researchers with the tools to analyze more complex signals and systems, ensuring robust and efficient digital designs. Whether in digital filtering, control system stability, or signal reconstruction, the principles outlined here remain integral.

References

- Oppenheim, A. V., Willsky, A. S., & Nawab, S. H. (1997). Signals and Systems. Prentice

Hall.
- Proakis, J. G., & Manolakis, D. G. (2006). Digital Signal Processing: Principles, Algorithms, and Applications. Pearson Education.
- Rabiner, L. R., & Gold, B. (1975). Theory and Application of Digital Signal Processing. Prentice Hall.

Note: This article aims to provide a comprehensive understanding of the Z-transform of a constant sampled function. For practical applications, always consider the specific system context, sampling rates, and potential aliasing effects.

The Function F T T Is Sampled Every Interval Of T Find The Z Transform Of The Sampled Function

The Function F T T Is Sampled Every Interval Of T Find The Z Transform Of The Sampled Function

Related Articles

- 2. Seniority Refers ToA. Pressures On Members Of Congress By Special Interest GroupsB. The Number Of
- A Medical Office Treats Two Different Types Of Patients, New Patients And Existing Patients. The New
- (Tree Height) Define A New Class Named BSTWithHeight That Extends BST With The Following Method: /**

Back to Home