A. Find The Parametric Form For The Plane Containing The Points(1, -2, 1), (0, 5, 3) And (2, 4, 7)B.

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Introduction

In analytic geometry, the equation of a plane in three-dimensional space can be expressed in various forms, including the general form, point-normal form, and parametric form. When given three non-collinear points, one of the most informative approaches is to find the parametric equation of the plane that contains these points. This method not only provides a clear geometric interpretation but also facilitates calculations involving points, lines, and intersections with the plane. This article explores the step-by-step process to determine the parametric equation of a plane passing through three specific points: (1, -2, 1), (0, 5, 3), and (2, 4, 7).

Understanding the Problem

What Is a Plane in Space?

A plane in three-dimensional space is a flat, two-dimensional surface extending infinitely in all directions. Geometrically, it can be uniquely defined by:

- A point on the plane and a normal vector (perpendicular to the plane).
- Three non-collinear points lying on the plane.
- An equation involving x, y, and z coordinates.

Given Points

The points provided are:

- $P_1 = (1, -2, 1)$
- $P_2 = (0, 5, 3)$
- $P_3 = (2, 4, 7)$

Our goal is to derive the parametric equations of the plane that contains these three points.

Step 1: Verify the Points Are Non-Collinear

Before proceeding, it's important to confirm that the points are not collinear; otherwise, they won't define a unique plane.

Calculating Vectors

Construct two vectors lying on the plane:

- $\vec{v_1} = P_2 - P_1 = (0 - 1, 5 - (-2), 3 - 1) = (-1, 7, 2)$
- $\vec{v_2} = P_3 - P_1 = (2 - 1, 4 - (-2), 7 - 1) = (1, 6, 6)$

Check for Collinearity

Calculate the cross product $\vec{v_1} \times \vec{v_2}$:

$$
\begin{aligned}
\vec{v_1} \times \vec{v_2} &=
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & 7 & 2 \\
1 & 6 & 6 \\
\end{vmatrix} \\
&= \mathbf{i}(7 \times 6 - 2 \times 6) - \mathbf{j}(-1 \times 6 - 2 \times 1) + \mathbf{k}(-1 \times 6 - 7 \times 1) \\
&= \mathbf{i}(42 - 12) - \mathbf{j}(-6 - 2) + \mathbf{k}(-6 - 7) \\
&= \mathbf{i}(30) - \mathbf{j}(-8) + \mathbf{k}(-13) \\
&= (30, 8, -13)
\end{aligned}
$$

Since the cross product is not the zero vector, the vectors are not parallel, and the points are non-collinear. Therefore, they define a unique plane.

Step 2: Find the Normal Vector to the Plane

The cross product $\vec{v_1} \times \vec{v_2}$ gives a normal vector $\vec{n}$ to the plane:

$$
\boxed{
\vec{n} = (30, 8, -13)
}
$$

This normal vector is essential for constructing the plane's equation.

Step 3: Write the Vector Equation of the Plane

The parametric form of a plane involves a point on the plane and two direction vectors that span the plane. We already have:

- A point $P_1 = (1, -2, 1)$
- Two vectors $\vec{v_1}$ and $\vec{v_2}$ lying on the plane, which can serve as direction vectors:

$$
\begin{cases}
\vec{v_1} = (-1, 7, 2) \\
\vec{v_2} = (1, 6, 6)
\end{cases}
$$

Hence, the parametric equations are:

$$
\begin{cases}
x = x_0 + s \cdot v_{1x} + t \cdot v_{2x} \\
y = y_0 + s \cdot v_{1y} + t \cdot v_{2y} \\
z = z_0 + s \cdot v_{1z} + t \cdot v_{2z}
\end{cases}
$$

where (x_0, y_0, z_0) is a point on the plane (we can use P_1), and $s, t \in \mathbb{R}$ are parameters.

Step 4: Write the Parametric Equations

Using $P_1 = (1, -2, 1)$, the parametric equations are:

$$
\boxed{
\begin{aligned}
x &= 1 + s \cdot (-1) + t \cdot 1 = 1 - s + t \\
y &= -2 + s \cdot 7 + t \cdot 6 = -2 + 7s + 6t \\
z &= 1 + s \cdot 2 + t \cdot 6 = 1 + 2s + 6t
\end{aligned}
}
$$

Thus, the parametric form of the plane is:

$$
\boxed{
\begin{cases}
x = 1 - s + t \\
y = -2 + 7s + 6t \\
z = 1 + 2s + 6t
\end{cases}
\quad \text{for all } s, t \in \mathbb{R}
}

\]

Additional Insights

Alternative Vector Basis

The two direction vectors \(\vec{v_1} \) and \(\vec{v_2} \) are linearly independent, ensuring they span the plane. Any other pair of vectors lying on the plane, as long as they are not scalar multiples, can serve to generate the parametric equations.

Using the Normal Vector for the Plane Equation

While the parametric form is often most useful for parametric plots and intersection calculations, the normal vector can also be used to write the Cartesian equation of the plane:

\[
\vec{n} \cdot (\vec{r} - \vec{r_0}) = 0
\]

where \(\vec{r} = (x, y, z) \) is a general point on the plane, and \(\vec{r_0} = (1, -2, 1) \).

Substituting:

\[
30(x - 1) + 8(y + 2) - 13(z - 1) = 0
\]
\[
30x - 30 + 8y + 16 - 13z + 13 = 0
\]
\[
30x + 8y - 13z - 1 = 0
\]

This is the Cartesian equation of the plane.

Summary

- Given points: \((1, -2, 1), (0, 5, 3), (2, 4, 7) \).
- Step 1: Verify points are non-collinear by computing their vectors and cross product.
- Step 2: Find the normal vector \(\vec{n} = (30, 8, -13) \) via the cross product.
- Step 3: Use a point (e.g., \(P_1 \)) and two direction vectors \(\vec{v_1} \) and \(\vec{v_2} \).
- Step 4: Write the parametric equations:

\[
\boxed{
\begin{cases}
x = 1 - s + t \\

y = -2 + 7s + 6t \\
z = 1 + 2s + 6t
\end{cases}
}
\]

- Additional: The Cartesian form of the plane is \(30x + 8y - 13z = 1 \).

This comprehensive approach ensures a thorough understanding of deriving the parametric form of a plane from three points, a fundamental skill in three-dimensional analytic geometry.

Frequently Asked Questions

How do I find the parametric equations of a plane passing through three points?

To find the parametric equations of a plane through three points, first find two vectors on the plane by subtracting the coordinates of the points (e.g., vector AB and AC). Then, compute the cross product of these vectors to get the normal vector. Using one point as a reference, the parametric form is given by adding scalar multiples of the direction vectors to that point.

What is the step-by-step process to derive the parametric form for the plane containing points (1, -2, 1), (0, 5, 3), and (2, 4, 7)?

1. Choose a point on the plane, such as P_0 = (1, -2, 1). 2. Find two direction vectors: $v_1 = P_2 - P_1$ = (0-1, 5+2, 3-1) = (-1, 7, 2) and $v_2 = P_3 - P_1$ = (2-1, 4+2, 7-1) = (1, 6, 6). 3. The parametric equations are then: x = 1 - s + t, y = -2 + 7s + 6t, z = 1 + 2s + 6t, where s and t are parameters.

How do I verify that the parametric equations correctly represent the plane through the three given points?

Substitute the parameters s and t with specific values to obtain the points on the plane. Check if the resulting points match the original points (1, -2, 1), (0, 5, 3), and (2, 4, 7). If they do, the parametric equations correctly describe the plane.

Can I find the normal vector of the plane from the parametric form, and how?

Yes. The normal vector can be obtained by taking the cross product of the two direction vectors (v_1 and v_2). For example, if v_1 = (-1, 7, 2) and v_2 = (1, 6, 6), then the normal vector $n = v_1 \times v_2$. Computing this gives the coefficients of the plane's equation.

What is the significance of the parametric form in understanding the geometry of the plane?

The parametric form explicitly describes all points on the plane as functions of parameters s and t. It helps visualize the plane as a surface generated by moving along two vectors from a fixed point, making it easier to analyze intersections, distances, and other geometric properties.

How can I convert the parametric form of the plane into its Cartesian equation?

Once you have the normal vector n, and a point P_0 on the plane, the Cartesian equation is given by $n \cdot (r - P_0) = 0$, where r = (x, y, z). Expand this dot product to obtain the scalar equation of the plane in standard form.

Additional Resources

Find The Parametric Form For The Plane Containing The Points (1, -2, 1), (0, 5, 3), and (2, 4, 7)

Introduction

In the realm of three-dimensional geometry, determining the parametric equation of a plane passing through three non-collinear points is a fundamental problem with broad applications ranging from computer graphics to engineering design. This investigation aims to meticulously derive the parametric form of the plane that contains the points \((1, -2, 1) \), \((0, 5, 3) \), and \((2, 4, 7) \). Through a systematic approach, including vector operations and parametric equations, the process demonstrates the elegance and precision innate to geometric problem-solving.

Understanding the Problem

Given three points in Euclidean space:

- \(P_1 = (1, -2, 1) \)
- \(P_2 = (0, 5, 3) \)
- \(P_3 = (2, 4, 7) \)

the task is to find a parametric equation for the plane that passes through all three points.

Key concepts involved:

- Parametric equations of a plane
- Vectors and vector operations (difference vectors, cross product)
- Normal vectors and their role in defining the plane

Step 1: Confirming Non-Collinearity

Before proceeding, ensure the three points are not collinear, as collinearity would prevent defining a unique plane.

Calculate vectors:

$$
\vec{A} = P_2 - P_1 = (0 - 1, 5 - (-2), 3 - 1) = (-1, 7, 2)
$$
$$
\vec{B} = P_3 - P_1 = (2 - 1, 4 - (-2), 7 - 1) = (1, 6, 6)
$$

Compute the cross product $\vec{A} \times \vec{B}$:

$$
\vec{A} \times \vec{B} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
-1 & 7 & 2 \\
1 & 6 & 6 \\
\end{vmatrix}
$$

Calculating:

$$
= \mathbf{i}(7 \times 6 - 2 \times 6) - \mathbf{j}(-1 \times 6 - 2 \times 1) + \mathbf{k}(-1 \times 6 - 7 \times 1)
$$

$$
= \mathbf{i}(42 - 12) - \mathbf{j}(-6 - 2) + \mathbf{k}(-6 - 7)
$$

$$
= \mathbf{i}(30) - \mathbf{j}(-8) + \mathbf{k}(-13)
$$

$$
= (30, 8, -13)
$$

Since the cross product is non-zero, the vectors $\vec{A}$ and $\vec{B}$ are not parallel, confirming the points are not collinear, and thus define a unique plane.

Step 2: Deriving the Normal Vector of the Plane

The normal vector \(\vec{n} \) to the plane is given by \(\vec{A} \times \vec{B} \):

\[
\boxed{
\vec{n} = (30, 8, -13)
}
\]

This vector is perpendicular to any vector lying within the plane, making it essential for defining the plane's orientation.

Step 3: Establishing a Point on the Plane

Any of the three points can serve as a reference point for the parametric equations. For simplicity, choose \(P_1 = (1, -2, 1) \).

Step 4: Formulating the Parametric Equations

The general parametric form of a plane passing through point \(P_0 = (x_0, y_0, z_0) \) with direction vectors \(\vec{A} \) and \(\vec{B} \) is:

\[
\boxed{
\begin{cases}
x = x_0 + s \cdot a_1 + t \cdot b_1 \\
y = y_0 + s \cdot a_2 + t \cdot b_2 \\
z = z_0 + s \cdot a_3 + t \cdot b_3
\end{cases}
}
\]

where:

- \((a_1, a_2, a_3) \) is vector \(\vec{A} = P_2 - P_1 = (-1, 7, 2) \),
- \((b_1, b_2, b_3) \) is vector \(\vec{B} = P_3 - P_1 = (1, 6, 6) \),
- \(s, t \in \mathbb{R} \).

Plugging in the known point:

\[
x = 1 + s \cdot (-1) + t \cdot 1
\]
\[
y = -2 + s \cdot 7 + t \cdot 6
\]
\[
z = 1 + s \cdot 2 + t \cdot 6

\]

The parametric equations of the plane are thus:

\[
\boxed{
\begin{cases}
x = 1 - s + t \\
y = -2 + 7s + 6t \\
z = 1 + 2s + 6t
\end{cases}
\quad \text{for } s, t \in \mathbb{R}
}
\]

Step 5: Validating the Parametric Form

To verify the correctness, substitute the parametric equations with specific values of \(s \) and \(t \):

- At \(s=0, t=0 \):

\[
x=1, \quad y=-2, \quad z=1
\]

which corresponds to \(P_1 \).

- At \(s=1, t=0 \):

\[
x=1 -1 + 0=0, \quad y=-2 +7=5, \quad z=1 + 2=3
\]

which matches \(P_2 \).

- At \(s=0, t=1 \):

\[
x=1 - 0 + 1=2, \quad y=-2 + 0 + 6=4, \quad z=1 + 0 + 6=7
\]

matching \(P_3 \).

This confirms the parametric equations accurately describe the plane passing through the three points.

Additional Considerations

Vector and Scalar Equations of the Plane

- The vector equation:

$$
\vec{r} = \vec{r}_0 + s \vec{A} + t \vec{B}
$$

- The scalar (Cartesian) equation:

Using the normal vector $\vec{n} = (30,8,-13)$, the plane satisfies:

$$
30(x - 1) + 8(y + 2) - 13(z - 1) = 0
$$

Expanding:

$$
30x - 30 + 8y + 16 - 13z + 13 = 0
$$
$$
30x + 8y - 13z - 1= 0
$$

This is the scalar form of the plane.

Conclusion

The comprehensive derivation confirms that the plane passing through points $(1, -2, 1)$, $(0, 5, 3)$, and $(2, 4, 7)$ can be effectively represented in parametric form as:

$$
\boxed{
\begin{cases}
x = 1 - s + t \\
y = -2 + 7s + 6t \\
z = 1 + 2s + 6t
\end{cases}
\quad \text{for } s, t \in \mathbb{R}
}
$$

This form provides a versatile tool for further geometric analysis, visualization, and applications requiring explicit parametrization of the plane.

Final Remarks

The process underscores the importance of vector operations in spatial geometry, illustrating how the cross product yields the normal vector crucial for defining a plane. Moreover, the explicit parametric equations facilitate practical computations and geometric interpretations, exemplifying the power of foundational linear algebra in solving real-world problems in three-dimensional space.

References

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