# A Solid Of Constant Density Is Bounded Below By The Plane Z=0 , Above By The Cone Z=2r ,r=0 , And On

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Introduction

In the realm of multivariable calculus and three-dimensional geometry, understanding the properties of solids bounded by various surfaces is crucial. These studies not only deepen our grasp of spatial relationships but also have practical applications in physics, engineering, and computer graphics. One particularly interesting problem involves analyzing a solid of constant density confined within specific geometric boundaries—namely a plane and a cone. Such problems often serve as foundational exercises for calculating mass, center of mass, moments of inertia, and other physical properties.

In this article, we explore a three-dimensional solid bounded below by the plane $z=0$, above by the cone $z=2r$ (where $r=\sqrt{x^2 + y^2}$), and with the third boundary specified as "And On"—which we interpret as the boundary on the lateral surface of the cone itself. We will delve into the geometry of this solid, set up the relevant integrals for volume and mass, and analyze its properties in detail. This comprehensive discussion aims to provide clarity on how to approach such problems, utilizing coordinate transformations, symmetry considerations, and integral calculus techniques.

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Geometric Description of the Solid

The Bounding Surfaces

The solid described is bounded by three key surfaces:

- Lower boundary: The plane $z=0$. This is the xy-plane, extending infinitely in all directions, serving as the base of the solid.

- Upper boundary: The cone given by $z=2r$, where $r=\sqrt{x^2 + y^2}$. This cone opens upward with its vertex at the origin and expands outward as $r$ increases.

- Lateral boundary: The cone's surface itself, which encloses the volume from the base to the apex.

Visualizing the Solid

To visualize this solid:

- Imagine placing a circular disk at the base $z=0$.

- As you move upward along the z-axis, the radius of the cone increases linearly with $z$, given by $r = z/2$.

- The boundary surface is the lateral surface of the cone $z=2r$, which forms a conical shell extending from the origin upward.

This shape resembles a conical frustum starting at the base and tapering toward the apex.

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Mathematical Formulation of the Boundaries

Converting to Cylindrical Coordinates

Given the symmetry about the z-axis, cylindrical coordinates $(r, \theta, z)$ are most appropriate:

$$
\begin{cases}
x = r \cos \theta \\
y = r \sin \theta \\
z = z
\end{cases}
$$

The Jacobian determinant for the transformation is:

$$
dV = r \, dr \, d\theta \, dz
$$

Boundary Conditions in Cylindrical Coordinates

- Base plane: $z=0$

- Upper surface (cone): $z=2r$

- Lateral surface: $z=2r$ (this boundary is already represented as the upper surface)

- Radial bounds: For each $z$, the radius $r$ varies from 0 up to the maximum radius at height $z$, which is $r = z/2$.

- Angular bounds: Since the shape is rotationally symmetric, $\theta$ varies from 0 to $2\pi$.

---

Calculating the Volume of the Solid

The volume $V$ of the solid is the integral over the region $R$:

$$
V = \int_{0}^{2\pi} \int_{0}^{r_{max}(\theta)} \int_{z_{min}}^{z_{max}(r)} r \, dz \, dr \, d\theta
$$

Setting the Limits

- For a fixed $r$, $z$ ranges from 0 up to the cone surface $z=2r$.

- For the entire solid, $r$ varies from 0 up to the maximum radius at the top of the cone, which occurs at the maximum height $z_{max}$. Since the cone extends infinitely upward unless specified otherwise, but in practical problems, the cone is truncated at a certain height $z=h$. For the sake of this problem, unless otherwise specified, assume the cone extends to a finite height $z=h$.

Assumption: Let's assume the cone is truncated at height $z=h$, which corresponds to $r_{max} = h/2$.

Volume Integral

$$
V = \int_{0}^{2\pi} \int_{0}^{h/2} \int_{0}^{2r} r \, dz \, dr \, d\theta
$$

- Integrate with respect to $z$:

$$
\int_{0}^{2r} r \, dz = r \times (2r) = 2 r^2
$$

- Now, the volume integral reduces to:

$$
V = \int_{0}^{2\pi} \int_{0}^{h/2} 2 r^2 \, dr \, d\theta
$$

- Integrate with respect to $r$:

$$
\int_{0}^{h/2} 2 r^2 \, dr = 2 \times \frac{(h/2)^3}{3} = 2 \times \frac{h^3}{24} = \frac{h^3}{12}
$$

- Finally, integrate over $\theta$:

$$
V = \int_{0}^{2\pi} \frac{h^3}{12} \, d\theta = \frac{h^3}{12} \times 2\pi = \frac{\pi
$$

h^3}{6}
\]

Result:

$$
\boxed{
\textbf{Volume of the truncated cone} = V = \frac{\pi h^3}{6}
}
$$

Note: If the cone extends infinitely, the volume is unbounded. Usually, a finite height $h$ is specified to make the problem meaningful.

---

Calculating the Mass of the Solid

Assumptions

- The solid has constant density $\rho$ (a positive constant).

- The density $\rho$ remains uniform throughout the volume.

Expression for Mass

$$
M = \rho \times V
$$

Using the volume calculated above:

$$
M = \rho \times \frac{\pi h^3}{6}
$$

This formula provides the total mass of the solid given the density and height.

---

Center of Mass Calculation

Determining the center of mass $(\bar{x}, \bar{y}, \bar{z})$ involves integrating over the volume:

$$
\bar{x} = \frac{1}{M} \int_V x \, \rho \, dV, \quad
\bar{y} = \frac{1}{M} \int_V y \, \rho \, dV, \quad
\bar{z} = \frac{1}{M} \int_V z \, \rho \, dV
$$

Symmetry Considerations

- The solid is rotationally symmetric about the z-axis; thus:

$$
\bar{x} = 0, \quad \bar{y} = 0
$$

- The z-coordinate of the center of mass:

$$
\bar{z} = \frac{1}{M} \int_V z \, \rho \, dV
$$

Calculating $\bar{z}$

Expressed in cylindrical coordinates:

$$
dV = r \, dr \, d\theta \, dz
$$

So,

$$
\int_V z \, dV = \int_{0}^{2\pi} \int_{0}^{h/2} \int_{0}^{2r} z \, r \, dz \, dr \, d\theta
$$

- Integrate over $z$:

$$
\int_{0}^{2r} z \, r \, dz = r \times \frac{(2r)^2}{2} = r \times 2 r^2 = 2 r^3
$$

- Integrate over $r$:

$$
\int_{0}^{h/2} 2 r^3 \, dr = 2 \times \frac{(h/2)^4}{4} = 2 \times \frac{h^4}{64} = \frac{h^4}{32}
$$

- Integrate over $\theta$:

$$
\int_{0}^{2\pi} d\theta = 2\pi
$$

- Total numerator:

$$
\int_V z \, dV = 2\pi \times \frac{h^4}{32} = \frac{\pi h^4}{16}
$$

\]

- Recall the total mass:

\[
M = \rho \times \frac{\pi h^3}{6}
\]

Center of mass in z-direction:

\[
\bar{z} = \frac{1}{M} \times \rho \times \frac{\pi h^4}{16

# Frequently Asked Questions

## What is the shape of the solid bounded below by the plane z=0 and above by the cone z=2r?

The solid is a conical frustum extending from the plane $z=0$ up to the cone $z=2r$, with a constant density, forming a truncated cone shape.

## How do you set up the triple integral to find the mass of this solid with constant density?

Using cylindrical coordinates, the triple integral is set with $r$ from 0 to a maximum radius determined by the cone, $\theta$ from 0 to $2\pi$, and $z$ from 0 up to $2r$, integrating the density over these bounds.

## What are the limits of integration for r and z in cylindrical coordinates for this solid?

The limits are $r$ from 0 to $R$ (where $R$ is determined by the intersection of the cone and the plane) and $z$ from 0 to $2r$, with $\theta$ from 0 to $2\pi$.

## How does the equation of the cone z=2r influence the integration bounds?

The cone's equation $z=2r$ defines the relationship between $z$ and $r$, so for each $r$, $z$ varies from 0 up to $2r$, shaping the upper boundary of the solid.

## How would the calculation change if the density of the solid is not constant but varies with position?

You would replace the constant density with a function $\rho(r, \theta, z)$ in the integral, making the mass calculation dependent on the specific variation of density within the solid.

# Additional Resources

Understanding the Solid of Constant Density Bounded by the Plane Z=0 and the Cone Z=2r

When studying three-dimensional bodies in calculus and physics, the concept of solids of constant density is fundamental, especially in the context of volume integrals, mass calculations, and center of mass determinations. In this detailed exploration, we focus on a specific solid bounded below by the plane $z=0$ and above by the cone $z=2r$, where $r = \sqrt{x^2 + y^2}$. This geometric configuration offers rich insights into the application of cylindrical coordinates, volume computation, and physical interpretations.

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1. Geometric Description of the Solid

1.1. Boundaries of the Solid

- Bottom Boundary: The plane $z=0$, which is the $xy$-plane.
- Upper Boundary: The cone $z=2r$, where $r = \sqrt{x^2 + y^2}$; this is a circular cone opening upward.
- Lateral Boundary: The surface of the cone itself, described by $z = 2r$.

1.2. Visualizing the Solid

Imagine a cone rising from the origin, with its vertex at the point $(0,0,0)$. The cone's surface is characterized by the equation:

$$
z = 2r \quad \Rightarrow \quad r = \frac{z}{2}.
$$

The solid is the volume enclosed between the plane $z=0$ (the flat base) and this cone. It resembles a conical "cup," with the base lying flat on the $xy$-plane and the cone extending upward, tapering outward as $z$ increases.

---

2. Mathematical Formulation

2.1. Coordinate System Choice

Given the symmetry of the problem around the $z$-axis, cylindrical coordinates $(r, \theta, z)$ are most suitable:

$$
x = r \cos \theta,\quad y = r \sin \theta,\quad z = z.
$$

This simplifies the description of the boundaries and the volume element:

$$
dV = r\, dr\, d\theta\, dz.
$$

2.2. Boundaries in Cylindrical Coordinates

- Vertical bounds: $z$ varies from $0$ to the cone surface $z = 2r$.
- Radial bounds: For a fixed $z$, the maximum radius is determined by the cone equation $z = 2r \Rightarrow r = z/2$.
- Angular bounds: $\theta$ spans from $0$ to $2\pi$.

Important note: Since the cone is symmetric about the $z$-axis, the angular bounds are straightforward.

---

3. Computing the Volume of the Solid

3.1. Setup of the Volume Integral

The volume $V$ can be expressed as:

$$
V = \int_{\theta=0}^{2\pi} \int_{z=0}^{z_{\max}} \int_{r=0}^{r_{\max}(z)} r\, dr\, dz\, d\theta,
$$

where:

- $r_{\max}(z) = \frac{z}{2}$,
- $z_{\max}$ is the maximum height of the cone.

Since $r$ goes from $0$ to $z/2$ at each $z$, and $\theta$ from $0$ to $2\pi$, the limits are:

$$
V = \int_{0}^{2\pi} \int_{0}^{z_{\max}} \int_{0}^{z/2} r\, dr\, dz\, d\theta.
$$

The maximum height $z_{\max}$ occurs where the cone intersects the boundary of the solid. Since the cone extends upward indefinitely, but physically, the solid is bounded by the cone surface itself, the maximum $z$ is unbounded unless specified.

However, in typical problems, the solid is considered bounded above at a specific height or with a finite boundary. If not explicitly given, we may consider the entire cone extending to infinity, which results in an unbounded volume.

Assuming a finite height $Z$, say $Z$, the volume integral becomes:

$$
V = \int_{0}^{2\pi} \int_{0}^{Z} \int_{0}^{z/2} r\, dr\, dz\, d\theta.

\]

For the purpose of a complete example, let's assume a finite height $Z$, and then the total volume is:

$$
V = 2\pi \int_{0}^{Z} \left( \int_{0}^{z/2} r\, dr \right) dz.
$$

Calculating the inner integral:

$$
\int_{0}^{z/2} r\, dr = \frac{1}{2} \left( r^2 \right)_{0}^{z/2} = \frac{1}{2} \left( \frac{z^2}{4} \right) = \frac{z^2}{8}.
$$

Thus,

$$
V = 2\pi \int_{0}^{Z} \frac{z^2}{8} dz = \frac{\pi}{4} \int_{0}^{Z} z^2 dz = \frac{\pi}{4} \left( \frac{z^3}{3} \right)_{0}^{Z} = \frac{\pi}{12} Z^3.
$$

Final volume formula (for finite height $Z$):

$$
V = \frac{\pi}{12} Z^3.
$$

If the problem involves the entire cone extending to infinity, the volume is unbounded.

---

4. Density and Mass Calculations

4.1. Constant Density Assumption

Let the density $\rho$ be constant throughout the solid. This simplifies mass calculations:

$$
\text{Mass } M = \rho \times V.
$$

Using the volume computed previously:

$$
M = \rho \times \frac{\pi}{12} Z^3,
$$

for the bounded case up to height $Z$.

---

5. Center of Mass and Moments

5.1. Coordinates of the Center of Mass

The center of mass $(\bar{x}, \bar{y}, \bar{z})$ is given by:

$$
\bar{x} = \frac{1}{M} \iiint_{V} x \, \rho\, dV, \quad
\bar{y} = \frac{1}{M} \iiint_{V} y \, \rho\, dV, \quad
\bar{z} = \frac{1}{M} \iiint_{V} z \, \rho\, dV.
$$

Due to symmetry about the $z$-axis, $\bar{x} = \bar{y} = 0$. The main focus is on $\bar{z}$.

5.2. Computing $\bar{z}$

$$
\bar{z} = \frac{1}{M} \iiint_{V} z \, \rho\, dV.
$$

Expressed in cylindrical coordinates:

$$
\iiint_{V} z\, dV = \int_{0}^{2\pi} \int_{0}^{Z} \int_{0}^{z/2} z\, r\, dr\, dz\, d\theta.
$$

Inner integral:

$$
\int_{0}^{z/2} r\, dr = \frac{z^2}{8}.
$$

Thus:

$$
\iiint_{V} z\, dV = 2\pi \int_{0}^{Z} z \times \frac{z^2}{8} dz = \frac{\pi}{4} \int_{0}^{Z} z^3 dz = \frac{\pi}{4} \left( \frac{z^4}{4} \right)_{0}^{Z} = \frac{\pi}{16} Z^4.
$$

Finally,

$$
\bar{z} = \frac{\rho \times \frac{\pi}{16} Z^4}{\rho \times \frac{\pi}{12} Z^3} = \frac{\frac{\pi}{16} Z^4}{\frac{\pi}{12} Z^3} = \frac{1/16}{1/12} Z = \frac{12}{16} Z = \frac{3}{4} Z.
$$

Result:

$$
\boxed{
\bar{z} = \frac{3}{4} Z.
}
$$

This indicates the center of mass along the $z$-axis is located at $75\%$ of the height $Z$.

---

6. Physical Interpretations and Applications

6.1. Relevance in Physics

- Mass Distribution: For a body with constant density, the mass distribution's symmetry simplifies calculations.
- Center of Mass: Knowing $\bar{z}$ helps in understanding how the mass is distributed vertically,

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