A Tank Contains 100 L Of Pure Water. Brine That Contains 0.1 Kg Of Salt Per Liter Enters The Tank At

A Tank Contains 100 L Of Pure Water. Brine That Contains 0.1 Kg Of Salt Per Liter Enters The Tank At is a classic problem in the study of mixing processes and differential equations. This scenario involves understanding how the concentration of salt in the tank evolves over time as brine continuously flows into the tank, which initially contains only pure water. Such problems are common in chemical engineering, environmental science, and even in certain biological systems where mixing and flow are involved. In this article, we will explore the mathematical modeling of this problem, analyze the solution process, and discuss its real-world applications.

Understanding the Problem Setup

Initial Conditions

The problem begins with a tank that contains a fixed volume of pure water, specifically 100 liters. Since the water is pure, the initial amount of salt in the tank is zero. This sets the initial condition:
- Volume of water in tank: 100 liters
- Initial salt amount: 0 kg
- Initial salt concentration: 0 kg/l

Inflow of Brine

Brine, which is salty water, is entering the tank at a specified rate. In this problem:
- The inflow rate of brine: typically given (e.g., 10 liters per minute)
- Salt concentration of incoming brine: 0.1 kg/liter

The brine adds both volume and salt to the tank, changing the concentration over time.

Outflow Conditions

Often, the problem assumes that the mixture is well-stirred, meaning the concentration of salt in the tank is uniform at any instant. The outflow rate is usually equal to the inflow rate for constant volume, but sometimes it can differ, which affects the model.

Mathematical Modeling of Salt Concentration Over Time

Variables and Parameters

To analyze the problem, define the following:
- $V(t)$: volume of liquid in the tank at time t (liters)
- $Q(t)$: amount of salt in the tank at time t (kg)
- $c(t)$: concentration of salt in the tank at time t (kg/liter)

Given the initial conditions:
- $V(0) = 100$ liters
- $Q(0) = 0$ kg

Assuming the inflow rate r (liters per minute) is constant, and the outflow rate is equal to r, then:
- $V(t) = 100$ liters (constant volume)

The inflow of brine introduces salt at a rate:
- $r \times 0.1$ kg/min

The outflow removes liquid and salt at a rate proportional to the current concentration:
- $r \times c(t)$ kg/min

Formulating the Differential Equation

The rate of change of salt in the tank is given by:

$$
\frac{dQ}{dt} = \text{Salt in} - \text{Salt out}
$$

Where:
- Salt in per unit time: $r \times 0.1$ kg/min
- Salt out per unit time: $r \times c(t) = r \times \frac{Q(t)}{V}$

Since V is constant at 100 liters, the differential equation simplifies to:

$$
\frac{dQ}{dt} = r \times 0.1 - r \times \frac{Q(t)}{100}
$$

Or:

$$
\frac{dQ}{dt} + \frac{r}{100} Q(t) = r \times 0.1

\]

This is a first-order linear differential equation, which can be solved using integrating factors.

Solving the Differential Equation

General Solution

Rearranged:
\[
\frac{dQ}{dt} + P Q = R
\]
where:
- \(P = \frac{r}{100} \)
- \(R = r \times 0.1 \)

The integrating factor \(\mu(t) \) is:
\[
\mu(t) = e^{\int P dt} = e^{\frac{r}{100} t}
\]

Multiplying through by \(\mu(t) \):
\[
e^{\frac{r}{100} t} \frac{dQ}{dt} + e^{\frac{r}{100} t} \frac{r}{100} Q = r \times 0.1 e^{\frac{r}{100} t}
\]

The left side simplifies to:
\[
\frac{d}{dt} \left(Q e^{\frac{r}{100} t} \right) = r \times 0.1 e^{\frac{r}{100} t}
\]

Integrating both sides:
\[
Q e^{\frac{r}{100} t} = \int r \times 0.1 e^{\frac{r}{100} t} dt + C
\]

Calculating the integral:
\[
\int r \times 0.1 e^{\frac{r}{100} t} dt = 0.1 r \times \frac{100}{r} e^{\frac{r}{100} t} + C' = 10 e^{\frac{r}{100} t} + C'
\]

Therefore:

$$
Q e^{\frac{r}{100} t} = 10 e^{\frac{r}{100} t} + C
$$

Dividing through by the exponential:

$$
Q(t) = 10 + C e^{-\frac{r}{100} t}
$$

Applying initial conditions:
- At $t = 0$, $Q(0) = 0$:

$$
0 = 10 + C \Rightarrow C = -10
$$

Finally:

$$
Q(t) = 10 - 10 e^{-\frac{r}{100} t}
$$

The salt concentration in the tank is:

$$
c(t) = \frac{Q(t)}{V} = \frac{10 - 10 e^{-\frac{r}{100} t}}{100} = 0.1 - 0.1 e^{-\frac{r}{100} t}
$$

This formula describes how the salt concentration approaches 0.1 kg/liter over time.

Interpreting the Solution

Steady-State Concentration

As $t \to \infty$, $e^{-\frac{r}{100} t} \to 0$, so:

$$
c(t) \to 0.1 \text{ kg/l}
$$

which matches the salt concentration of incoming brine. The system reaches equilibrium where the salt inflow equals salt outflow.

Time to Reach Near Equilibrium

The rate at which the concentration approaches equilibrium depends on the flow rate r. For example, if $r = 10$ liters per minute:
- The exponential decay rate is $\frac{r}{100} = 0.1$
- After approximately 3 time constants ($3 \times \frac{100}{r}$), or 30 minutes, the concentration is very close to 0.1 kg/l.

Practical Applications of the Salt-Water Mixing Model

Chemical Engineering and Process Control

Understanding how concentrations change over time is essential for designing mixing tanks, chemical reactors, and water treatment facilities. The mathematical model helps engineers optimize flow rates to reach desired concentrations efficiently.

Environmental Science and Water Quality

In environmental management, such models predict how pollutants disperse in water bodies, aiding in pollution control and remediation efforts.

Biological Systems and Medicine

Similar principles apply to how nutrients or drugs disperse within biological tissues or fluids, influencing dosing and treatment strategies.

Extensions and Variations of the Model

Variable Inflow and Outflow Rates

Instead of constant rates, models can incorporate changing flow rates, which lead to more complex differential equations solvable via numerical methods.

Multiple Inflows or Outflows

Adding multiple sources or sinks introduces additional terms, making the analysis more intricate but more representative of real-world systems.

Non-Constant Volume Systems

If the volume of the tank isn't fixed, the differential equations become nonlinear, requiring advanced techniques such as numerical simulation to solve.

Conclusion

The problem of a tank containing pure water into which salty brine is introduced exemplifies the fundamental principles of mixing and differential equations. By setting up and solving the appropriate differential equations, one can predict how the concentration of salt evolves over time, reach steady-state conditions, and optimize process parameters. These models are invaluable across various fields, from chemical engineering to environmental science, providing insights that inform practical decision-making. Whether designing industrial processes or understanding natural systems, mastering such models enables better control and understanding of dynamic mixing phenomena.

Frequently Asked Questions

How much salt is initially in the tank when it contains 100 liters of pure water?

Initially, there is no salt in the tank since it contains only pure water, so the initial amount of salt is 0 kg.

If brine containing 0.1 kg of salt per liter enters the tank, how much salt is added per liter of inflow?

The inflowing brine adds 0.1 kg of salt for every liter that enters the tank.

What is the rate at which salt concentration in the tank changes over time with continuous inflow and outflow?

The rate of change depends on the inflow of salt with the incoming brine and the outflow of salt with the mixture leaving the tank, which can be modeled using differential equations considering flow rate and current salt concentration.

How do you determine the steady-state salt concentration in the tank?

The steady-state salt concentration is found by setting the rate of change of salt to zero and solving for the concentration, which balances the inflow of salt with the outflow of salt at equilibrium.

If the inflow rate is 10 liters per minute, how long will it take for the salt concentration in the tank to reach 50% of its maximum value?

This requires solving the differential equation governing the system; generally, the time to reach a certain concentration can be found using exponential decay formulas based on flow rate and initial conditions.

What practical applications can this mixing problem illustrate in real-world scenarios?

It illustrates concepts in chemical engineering, water treatment, and environmental science, such as dilution processes, tank mixing dynamics, and pollutant dispersal modeling.

Additional Resources

A tank contains 100 liters of pure water. Brine that contains 0.1 kg of salt per liter enters the tank at a specified rate, creating a dynamic system where salt concentration changes over time. This scenario is a classic example of a mixing problem in differential equations, often encountered in chemical engineering, environmental science, and related fields. Understanding how the salt concentration evolves requires a detailed analysis of the inflow, outflow, and the resulting change in the tank's salt content.

In this article, we will delve into the mathematical modeling of this problem, explore the differential equations involved, and interpret the solutions to gain insights into how the salt concentration varies over time. Our goal is to present this complex topic in a clear, accessible manner suitable for readers with a basic understanding of calculus and differential equations, while maintaining technical rigor.

The Setup: Initial Conditions and Parameters

The Initial State

The tank initially contains 100 liters of pure water, which implies that at time \(t = 0 \), the amount of salt in the tank is zero. This initial condition simplifies the initial state of the system:

- Volume of water in tank: 100 liters
- Initial salt amount: 0 kg

The Inflow of Brine

The brine enters the tank at a constant rate r liters per unit time. The concentration of salt in this inflow is 0.1 kg/liter, meaning each liter of inflow brings in 0.1 kg of salt.

- Inflow rate: r liters per unit time
- Inflow salt concentration: 0.1 kg/liter

The total salt entering the tank per unit time is thus:

$$
\text{Salt inflow} = 0.1 \times r \quad \text{kg per unit time}
$$

The Outflow

The mixture in the tank flows out at the same rate r liters per unit time, maintaining a constant volume of 100 liters. The outflow contains salt proportionally to the concentration in the tank at any given time.

- Outflow rate: r liters per unit time
- Concentration of salt in the tank at time t: $c(t)$ kg/liter

The amount of salt leaving per unit time is:

$$
\text{Salt outflow} = c(t) \times r
$$

Modeling the System: Differential Equation Formulation

The Key Variables

Let:

- $Q(t)$ = the total amount of salt in the tank at time t, in kilograms.
- V = volume of water in the tank = 100 liters (constant).

Since the volume remains constant, the concentration of salt in the tank at time t is:

\[

c(t) = \frac{Q(t)}{V} = \frac{Q(t)}{100}
\]

The Rate of Change of Salt

The net rate of change of salt in the tank is the difference between the inflow and outflow:

\[
\frac{dQ}{dt} = \text{Salt inflow} - \text{Salt outflow}
\]

Substituting the expressions:

\[
\frac{dQ}{dt} = 0.1 r - c(t) r = 0.1 r - \frac{Q(t)}{100} r
\]

Simplify:

\[
\frac{dQ}{dt} = 0.1 r - \frac{r}{100} Q(t)
\]

This is a linear differential equation in \(Q(t) \):

\[
\boxed{
\frac{dQ}{dt} + \frac{r}{100} Q(t) = 0.1 r
}
\]

Solving the Differential Equation

Standard Form and Integrating Factor

The differential equation:

\[
\frac{dQ}{dt} + p Q = q
\]

where:

- $p = \frac{r}{100}$
- $q = 0.1 r$

The integrating factor $\mu(t)$ is:

$$
\mu(t) = e^{\int p dt} = e^{\frac{r}{100} t}
$$

Multiplying through by $\mu(t)$:

$$
e^{\frac{r}{100} t} \frac{dQ}{dt} + e^{\frac{r}{100} t} \frac{r}{100} Q = 0.1 r e^{\frac{r}{100} t}
$$

which simplifies to:

$$
\frac{d}{dt} \left(e^{\frac{r}{100} t} Q \right) = 0.1 r e^{\frac{r}{100} t}
$$

Integrate both sides with respect to t:

$$
e^{\frac{r}{100} t} Q = \int 0.1 r e^{\frac{r}{100} t} dt + C
$$

Computing the Integral

$$
\int 0.1 r e^{\frac{r}{100} t} dt = 0.1 r \times \frac{100}{r} e^{\frac{r}{100} t} + C' = 10 e^{\frac{r}{100} t} + C'
$$

So the general solution is:

$$
e^{\frac{r}{100} t} Q = 10 e^{\frac{r}{100} t} + C
$$

Dividing both sides by $e^{\frac{r}{100} t}$:

$$
Q(t) = 10 + C e^{-\frac{r}{100} t}
$$

Applying the Initial Condition

At $t = 0$, $Q(0) = 0$:

$$
0 = 10 + C \times e^{0} \Rightarrow C = -10
$$

Therefore, the particular solution:

$$
\boxed{
Q(t) = 10 - 10 e^{-\frac{r}{100} t}
}
$$

Interpreting the Solution: Salt Concentration Over Time

Recall the concentration:

$$
c(t) = \frac{Q(t)}{100} = \frac{10 - 10 e^{-\frac{r}{100} t}}{100} = 0.1 - 0.1 e^{-\frac{r}{100} t}
$$

This expression describes how the salt concentration in the tank evolves over time, approaching a steady-state value.

Steady-State Salt Concentration

As $t \to \infty$, the exponential term tends to zero:

$$
c(t) \to 0.1 - 0 = 0.1 \text{ kg/liter}
$$

This aligns with physical intuition: in the long run, the tank's salt concentration stabilizes at the same level as the inflow brine, since the inflow and outflow are balanced.

Practical Applications and Implications

Rate of Approach to Equilibrium

The exponential decay term $e^{-\frac{r}{100} t}$ determines how quickly the system approaches steady state. The larger the inflow rate r:

- The faster the exponential term diminishes.
- The quicker the concentration reaches near its steady-state value.

Time to Reach a Certain Concentration

Suppose we want to find the time t when the concentration reaches a certain level $c^\wedge$:

$$
c^\wedge = 0.1 - 0.1 e^{-\frac{r}{100} t}
$$

Rearranged:

$$
e^{-\frac{r}{100} t} = 1 - \frac{c^\wedge}{0.1}
$$

$$
t = -\frac{100}{r} \ln \left(1 - \frac{c^\wedge}{0.1} \right)
$$

This formula allows calculation of the time needed to reach a desired salt concentration, given the inflow rate.

Effect of Different Inflow Rates

- Higher r: Faster approach to equilibrium.
- Lower r: Slower change, longer time to reach steady state.

Impact on Environmental and Industrial Processes

Understanding such mixing dynamics assists in designing chemical reactors, water treatment facilities, and environmental remediation strategies. Controlling inflow rates can optimize the speed and efficiency of achieving desired concentrations, minimize waste, and ensure safety standards.

Considerations for Real-World Scenarios

While the mathematical model provides clear insights, real-world applications often involve additional complexities:

- Variable inflow rates: In practice, inflow may fluctuate.
- Non-constant volume: Evaporation or other factors can alter volume.
- Non-uniform mixing: Complete mixing assumptions may not hold immediately.
- Chemical reactions: Salt may precipitate or react, altering concentrations.

Engineers and scientists incorporate these factors into more sophisticated models, but the core principles outlined here serve as foundational concepts.

Summary and Key Takeaways

- The problem involves a classic mixing process modeled by a first-order linear differential equation.
- The solution provides an explicit formula for the salt amount \(Q(t) \) and concentration \(c(t) \) over time.
- The system approaches a steady state where the salt concentration equals

A Tank Contains 100 L Of Pure Water Brine That Contains 0 1 Kg Of Salt Per Liter Enters The Tank At

A Tank Contains 100 L Of Pure Water Brine That Contains 0 1 Kg Of Salt Per Liter Enters The Tank At

Related Articles

- A Study By Consumer Reports Showed That 64% Of Supermarket Shoppers Believe Supermarket Brands To Be
- Amanda Rented A Bike From Ted's Bikes.It Costs $10 For The Helmet Plus $4.25 Per Hour.If Amanda Paid
- Amazons Efforts To Offer Customers The Option To Return Or Purchase Items In The Physical World Is A

Back to Home