Algebra Find The Eigenvalues, And Give Bases For The Eigenspaces Of The Following 4 X 4 Matrix: [2 2

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Understanding eigenvalues and eigenspaces is a fundamental aspect of linear algebra, especially when analyzing complex matrices. They provide insight into the matrix's structure, its transformations, and applications across various scientific and engineering fields. In this comprehensive guide, we will explore how to find the eigenvalues of a 4x4 matrix, specifically the matrix [2 2; ...], determine the eigenspaces, and identify bases for these eigenspaces. Whether you're a student preparing for exams or a professional dealing with matrix computations, this detailed explanation aims to clarify each step involved in the process.

Introduction to Eigenvalues and Eigenspaces

Before delving into calculations, it is essential to understand what eigenvalues and eigenspaces are and their significance.

Eigenvalues

- An eigenvalue of a matrix A is a scalar λ such that there exists a non-zero vector v satisfying A v = λ v.
- Geometrically, eigenvalues represent factors by which the eigenvectors are scaled during the transformation described by A.

- They are solutions to the characteristic polynomial det(A - λ I) = 0, where I is the identity matrix.

Eigenspaces

- The eigenspace corresponding to an eigenvalue λ is the set of all eigenvectors associated with λ, along with the zero vector.
- It is a subspace of the vector space and can be found by solving (A - λ I) v = 0.
- The basis for the eigenspace provides a minimal set of vectors that span this subspace.

Given Matrix and Its Structure

The matrix in question is:

$$
A = \begin{bmatrix}
2 & 2 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 1 \\
0 & 0 & 0 & 3 \\
\end{bmatrix}
$$

This matrix is block upper-triangular with two blocks:
- A 2x2 block: $\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$
- A 2x2 block: $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

Analyzing such matrices simplifies the process of finding eigenvalues because the eigenvalues of a block upper-triangular matrix are the union of the eigenvalues of its diagonal blocks.

Step 1: Find the Eigenvalues

The eigenvalues are derived from the characteristic polynomial:

$$
\det(A - \lambda I) = 0
$$

Given the block structure, the characteristic polynomial is the product of the determinants of the blocks:

$$
\det(A - \lambda I) = \det \left(\begin{bmatrix}
2 - \lambda & 2 \\
0 & 2 - \lambda \\
\end{bmatrix} \right) \times \det \left(\begin{bmatrix}
3 - \lambda & 1 \\
0 & 3 - \lambda \\
\end{bmatrix} \right)
$$

Calculating each determinant:

1. For the first block:

$$
\det \left(\begin{bmatrix}
2 - \lambda & 2 \\
0 & 2 - \lambda \\
\end{bmatrix} \right) = (2 - \lambda)^2
$$

2. For the second block:

$$
\det \left(\begin{bmatrix}
3 - \lambda & 1 \\
0 & 3 - \lambda \\
\end{bmatrix} \right) = (3 - \lambda)^2
$$

Thus, the characteristic polynomial is:

$$
(2 - \lambda)^2 \times (3 - \lambda)^2 = 0
$$

The eigenvalues are solutions to this polynomial:

$$
\boxed{
\lambda = 2 \quad \text{(multiplicity 2)}, \quad \lambda = 3 \quad \text{(multiplicity 2)}
}
$$

Summary of Eigenvalues:

- $\lambda_1 = 2$, with algebraic multiplicity 2
- $\lambda_2 = 3$, with algebraic multiplicity 2

Next, we proceed to find the eigenspaces corresponding to each eigenvalue.

Step 2: Find the Eigenspaces and Bases

For each eigenvalue, solve the system $(A - \lambda I) v = 0$ to find the eigenspace.

Eigenvalue $\lambda = 2$

Compute $(A - 2 I)$:

$$
A - 2 I = \begin{bmatrix}
0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

Set up the homogeneous system:

$$
(A - 2 I) v = 0
$$

Let $v = (v_1, v_2, v_3, v_4)^T$.

The equations:

1. $0 \cdot v_1 + 2 v_2 + 0 + 0 = 0 \Rightarrow 2 v_2 = 0 \Rightarrow v_2 = 0$

2. $0 = 0$ (second row, no constraint)

3. $0 \cdot v_1 + 0 + 1 v_3 + 1 v_4 = 0 \Rightarrow v_3 + v_4 = 0$

4. $0 \cdot v_1 + 0 + 0 + 1 v_4 = 0 \Rightarrow v_4 = 0$

From equation 4: $v_4 = 0$

From equation 3: $v_3 + 0 = 0 \Rightarrow v_3 = 0$

From equation 1: $v_2 = 0$

v_1 is free (since it does not appear in any equations).

Thus, the general solution:

$$
v = (v_1, 0, 0, 0)^T
$$

where v_1 is arbitrary.

Basis for eigenspace corresponding to $\lambda=2$:

$$
\boxed{
\text{Eigenspace } E_2 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}
}
$$

Dimension: 1 (geometric multiplicity)

Eigenvalue $\lambda = 3$

Compute $(A - 3 I)$:

$$
A - 3 I = \begin{bmatrix}
-1 & 2 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Solve:

$$
(A - 3 I) v = 0
$$

Let $v = (v_1, v_2, v_3, v_4)^T$.

Equations:

1. $-1 v_1 + 2 v_2 = 0 \Rightarrow 2 v_2 = v_1$

2. $-1 v_2 = 0 \Rightarrow v_2 = 0$

3. $0 v_3 + 1 v_4 = 0 \Rightarrow v_4 = 0$

4. $0 = 0$ (no new info)

From (2): $v_2 = 0$

From (1): $v_1 = 2 v_2 = 0$

From (3): $v_4 = 0$

v_3 is free.

Summary:

$$
v = (0, 0, v_3, 0)^T
$$

Eigenvectors are scalar multiples of $(0, 0, 1, 0)^T$.

Basis for eigenspace corresponding to $\lambda=3$:

$$
\boxed{
E_3 = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}
}
$$

Dimension: 1 (geometric multiplicity)

Summary of Results

Eigenvalue	Algebraic Multiplicity	Geometric Multiplicity	Basis for Eigenspace
2	2	1	\(\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0

Frequently Asked Questions

What is the first step in finding the eigenvalues of a 4x4 matrix?

The first step is to subtract λ times the identity matrix from the matrix and compute the characteristic polynomial by finding the determinant of this matrix, then solve for λ.

How do you compute the characteristic polynomial of a 4x4 matrix?

You compute the determinant of (A - λI), where A is your matrix, and set it equal to zero. Expanding this determinant gives a polynomial in λ whose roots are the eigenvalues.

What is the significance of the eigenvalues in the context of a 4x4 matrix?

Eigenvalues indicate the scalar factors by which eigenvectors are scaled during the linear transformation represented by the matrix, revealing key properties like stability and diagonalizability.

How do you find eigenvectors corresponding to each eigenvalue?

For each eigenvalue λ, solve the homogeneous system (A - λI) x = 0 to find the eigenvectors forming the eigenspace.

What is a basis for an eigenspace, and how is it found?

A basis for an eigenspace consists of linearly independent eigenvectors associated with a particular eigenvalue. It is found by solving (A - □I) x = 0 for that □.

Can you give an example of how to find eigenvalues for a matrix starting with [2 2 ...]?

Yes. For a matrix starting with [2 2 ...], you first write the full matrix, compute the characteristic polynomial det(A - □I), and solve for □ to find the eigenvalues.

How do algebraic and geometric multiplicities relate to eigenvalues and eigenspaces?

The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial, while the geometric multiplicity is the dimension of its eigenspace. They can be equal or differ, affecting diagonalizability.

What tools or methods can simplify finding eigenvalues and eigenspaces of a 4x4 matrix?

Using row operations, characteristic polynomial calculations, and software tools like MATLAB or Python (NumPy, SymPy) can streamline the process.

Why is it important to find bases for eigenspaces in linear algebra?

Finding bases for eigenspaces helps in understanding the structure of the matrix, diagonalization, and solving systems of differential equations, among other applications.

Given a matrix starting with [2 2 ...], how do you interpret the

eigenvalues once found?

Eigenvalues reveal the scaling factors for eigenvectors under the matrix transformation, indicating properties like expansion, contraction, or rotation in the space.

Additional Resources

Algebra Find The Eigenvalues, And Give Bases For The Eigenspaces Of The Following 4 X 4 Matrix: [2 2

Understanding the intricate world of linear algebra often begins with the fundamental concepts of eigenvalues and eigenvectors. These mathematical tools provide insights into the behavior of linear transformations, stability analysis, and many applications across engineering, physics, computer science, and data analysis. Today, we delve into a specific problem: determining the eigenvalues and bases for the eigenspaces of a given 4x4 matrix. This exercise not only reinforces core algebraic principles but also illustrates the practical steps involved in analyzing complex matrices.

Introduction to Eigenvalues and Eigenvectors

Before tackling the problem, it’s essential to understand what eigenvalues and eigenvectors are, why they matter, and how they relate to matrices.

Eigenvalues are scalars associated with a matrix that characterize how the matrix stretches or compresses certain vectors in space. When a matrix acts on an eigenvector, the output vector points in the same direction as the original eigenvector, scaled by the eigenvalue.

Eigenvectors, on the other hand, are non-zero vectors that only change in magnitude (not direction) when transformed by the matrix. Finding eigenvalues and eigenvectors helps us diagonalize matrices, simplify matrix powers, and analyze systems of differential equations.

The general equation governing these concepts is:

$$A \mathbf{v} = \lambda \mathbf{v}$$

where A is the matrix, $\mathbf{v}$ is an eigenvector, and λ is the associated eigenvalue.

The Matrix in Question

The matrix given in the problem appears incomplete in the prompt: "[2 2". For the purpose of this analysis, let’s assume a typical 4x4 matrix structure that is common in such exercises. Suppose the matrix is:

$$
A = \begin{bmatrix}
2 & 2 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 1 \\
0 & 0 & 0 & 3 \\
\end{bmatrix}
$$

This matrix combines blocks that are upper triangular, which simplifies eigenvalue calculations. If the actual matrix differs, the methodology remains similar; keys are to find roots of the characteristic polynomial and then to determine the eigenspaces.

Step 1: Find the Eigenvalues

The eigenvalues are solutions to the characteristic equation:

\[

\det(A - \lambda I) = 0

\]

where \(I \) is the identity matrix of size 4x4, and \(\det \) denotes the determinant.

Constructing \(A - \lambda I \):

\[

A - \lambda I = \begin{bmatrix}

2 - \lambda & 2 & 0 & 0 \\

0 & 2 - \lambda & 0 & 0 \\

0 & 0 & 3 - \lambda & 1 \\

0 & 0 & 0 & 3 - \lambda \\

\end{bmatrix}

\]

Since the matrix is block upper triangular, its determinant is the product of the determinants of the blocks:

\[

\det(A - \lambda I) = \det \begin{bmatrix}

2 - \lambda & 2 \\

0 & 2 - \lambda \\

\end{bmatrix} \times \det \begin{bmatrix}

3 - \lambda & 1 \\

0 & 3 - \lambda \\

\end{bmatrix}
\]

Calculating each:

- First block:

\[
\det \begin{bmatrix}
2 - \lambda & 2 \\
0 & 2 - \lambda \\
\end{bmatrix} = (2 - \lambda)^2
\]

- Second block:

\[
\det \begin{bmatrix}
3 - \lambda & 1 \\
0 & 3 - \lambda \\
\end{bmatrix} = (3 - \lambda)^2
\]

Thus, the characteristic polynomial:

\[
\det(A - \lambda I) = (2 - \lambda)^2 \times (3 - \lambda)^2
\]

Setting this equal to zero gives the eigenvalues:

$$
(2 - \lambda)^2 (3 - \lambda)^2 = 0
$$

which implies:

$$
\boxed{
\lambda = 2 \quad \text{(with multiplicity 2)} \\
\lambda = 3 \quad \text{(with multiplicity 2)}
}
$$

Summary:

- Eigenvalue $\lambda = 2$ (multiplicity 2)
- Eigenvalue $\lambda = 3$ (multiplicity 2)

Step 2: Find the Eigenspaces and Their Bases

With the eigenvalues established, the next step is to determine the eigenspaces—sets of all eigenvectors corresponding to each eigenvalue—and to find bases for these spaces.

Eigenspace for $\lambda = 2$

We seek solutions to:

\[

(A - 2I) \mathbf{v} = 0

\]

Calculating \(A - 2I \):

\[

A - 2I = \begin{bmatrix}

0 & 2 & 0 & 0 \\

0 & 0 & 0 & 0 \\

0 & 0 & 1 & 1 \\

0 & 0 & 0 & 1 \\

\end{bmatrix}

\]

The system translates into:

\[

\begin{cases}

0 \cdot v_1 + 2 v_2 + 0 \cdot v_3 + 0 \cdot v_4 = 0 \\

0 = 0 \\

0 \cdot v_1 + 0 \cdot v_2 + 1 v_3 + 1 v_4 = 0 \\

0 = 0 \\

\end{cases}

\]

From the second and fourth equations, which are trivial, focus on the first and third:

- First:

\[

2 v_2 = 0 \Rightarrow v_2 = 0

- Third:

$$
v_3 + v_4 = 0 \Rightarrow v_4 = -v_3
$$

The free variables are v_1 and v_3:

- v_1 is free (no constraints)
- v_3 is free, with $v_4 = -v_3$

Expressed as:

$$
\mathbf{v} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}
$$

Basis for E_2:

$$
\left\{
\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},
\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}
\right\}
$$

Eigenspace for $\lambda = 3$

Similarly, solve:

$$
(A - 3I) \mathbf{v} = 0
$$

Calculate $A - 3I$:

$$
A - 3I = \begin{bmatrix}
-1 & 2 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

Set up the system:

$$
\begin{cases}
-1 v_1 + 2 v_2 = 0 \\
-1 v_2 = 0 \\
0 v_3 + 1 v_4 = 0 \\
0 = 0 \\
\end{cases}
$$

From the second:

$$
- v_2 = 0 \Rightarrow v_2 = 0
$$

From the first:

$$
- v_1 + 2 \cdot 0 = 0 \Rightarrow v_1 = 0
$$

From the third:

$$
v_4 = 0
$$

Variables v_3 is free:

$$
\mathbf{v} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
$$

Basis for E_3:

$$
\left\{
\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}
\right\}
$$

This eigenspace is one-dimensional, matching the algebraic multiplicity of eigenvalue 3, confirming its geometric multiplicity.

Implications and Applications of Eigenanalysis

Understanding the eigenvalues and eigenspaces of matrices like the one examined above has profound implications across various fields:

- Diagonalization: The ability to find a basis of eigenvectors allows the matrix to be diagonalized, simplifying many computations, especially matrix powers and exponentials.
- Stability Analysis:

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