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Solving systems of differential equations is a fundamental aspect of applied mathematics, physics, engineering, and many scientific disciplines. When dealing with nonhomogeneous systems—those with external forcing functions or input terms—finding particular solutions becomes essential to understanding the complete behavior of the system. One powerful technique for constructing these particular solutions is the Method of Undetermined Coefficients. This method involves guessing the form of the particular solution based on the nature of the nonhomogeneous terms and then determining unknown coefficients through substitution.

In this comprehensive guide, we will explore how to apply the method of undetermined coefficients to find particular solutions for systems of differential equations. We will cover the theoretical foundation, step-by-step procedures, examples, and tips to master this technique, ensuring you can confidently solve various types of systems.

Understanding the Method of Undetermined Coefficients

Before diving into the application process, it is important to understand what the method entails and when it is appropriate to use.

What Is the Method of Undetermined Coefficients?

The method involves:

- Making an educated guess (ansatz) for the form of the particular solution based on the nonhomogeneous terms.
- Introducing unknown coefficients into this guess.
- Substituting the guessed solution into the original differential system.
- Solving for these coefficients by equating coefficients of like terms.

This approach simplifies the process of finding particular solutions, especially when the nonhomogeneous part is a linear combination of exponential, polynomial, sine, or cosine functions.

When to Use the Method

The method is most effective under the following conditions:

- The nonhomogeneous term(s) are of specific types, such as:
- Polynomials
- Exponentials
- Sine and cosine functions
- The differential system is linear with constant coefficients.

It is not suitable for systems with non-constant coefficients, transcendental functions, or non-linear terms.

Step-by-Step Procedure to Apply the Method

Applying the method involves systematic steps. Below is a structured approach:

Step 1: Write Down the System

Identify the system of differential equations in matrix form:

$$\mathbf{Y'}(t) = A \mathbf{Y}(t) + \mathbf{G}(t)$$

where:

- $\mathbf{Y}(t)$ is the vector of unknown functions.
- A is a constant coefficient matrix.
- $\mathbf{G}(t)$ is the nonhomogeneous (forcing) vector.

Step 2: Find the Complementary (Homogeneous) Solution

Solve the homogeneous system:

$$\mathbf{Y'}(t) = A \mathbf{Y}(t)$$

to determine the general solution $\mathbf{Y}_h(t)$. This involves:

- Finding the eigenvalues and eigenvectors of A.
- Constructing the general homogeneous solution.

Step 3: Analyze the Nonhomogeneous Term $\mathbf{G}(t)$

Identify the form of the forcing function:

- Is it a polynomial? (e.g., t^n)
- An exponential? (e.g., e^{kt})
- Trigonometric functions? (e.g., $\sin \omega t, \cos \omega t$)
- A combination of these types.

This analysis guides the choice of the form of the particular solution.

Step 4: Make an Initial Guess for the Particular Solution

Based on the nature of $\mathbf{G}(t)$, assume a form for $\mathbf{Y}_p(t)$. For example:

- For polynomial forcing terms, assume polynomial solutions.

- For exponential forcing, assume exponential solutions.
- For sinusoidal forcing, assume sine and cosine functions.

Note: If any guessed form overlaps with the homogeneous solution, multiply by t to account for duplication (resonance).

Step 5: Introduce Unknown Coefficients

Assign coefficients to the terms in the guessed particular solution. For example:

$$\mathbf{Y}_p(t) = \mathbf{P}(t) = \text{(polynomial with unknown coefficients)}$$

or

$$\mathbf{Y}_p(t) = \mathbf{E}(t) = \text{(exponential with unknown coefficients)}$$

or a combination of these.

Step 6: Substitute into the Original System

Compute derivatives of $\mathbf{Y}_p(t)$, substitute into the original differential equations, and expand.

Step 7: Equate Coefficients and Solve for Unknowns

Match coefficients of corresponding terms in the resulting equations to determine the unknown coefficients.

Step 8: Write the General Solution

Combine the homogeneous and particular solutions:

$$\mathbf{Y}(t) = \mathbf{Y}_h(t) + \mathbf{Y}_p(t)$$

and interpret the complete solution.

Example: Applying the Method to a System of Differential Equations

Let's illustrate the process with a concrete example.

Given System:

$$
\begin{cases}
x' = 2x + y + 3e^{2t} \\
y' = -x + 4y
\end{cases}
$$

Our goal is to find a particular solution $(x_p(t), y_p(t))$.

Step 1: Homogeneous System