Calculate The Laplace Transform L{f(t)} For The Function F(t) = (1 3te-t 4te-3t)2 And Then Determine

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Introduction to the Laplace Transform and Its Significance

The Laplace transform is a powerful integral transform widely used in engineering, physics, and mathematics to simplify the process of solving differential equations. It converts complex differential equations in the time domain into algebraic equations in the s-domain, making the analysis and solution process more manageable.

Specifically, the Laplace transform of a function $f(t)$, denoted as $L\{f(t)\}$, is defined as:

$$
L\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \, dt
$$

where s is a complex frequency parameter.

Understanding how to compute the Laplace transform for various functions, especially composite and complex expressions, is essential for analysts and engineers. This guide will walk you through calculating the Laplace transform of a specific function involving exponential and polynomial terms, followed by determining the explicit form of the transform.

Overview of the Function $F(t)$

The function under consideration is:

$$
F(t) = \left(1 + 3t e^{-t} + 4t e^{-3t} \right)^2
$$

This function combines polynomial terms with exponential functions, making the Laplace transform calculation intricate but approachable using systematic techniques such as expansion, linearity, and known transforms.

Key features of $F(t)$:

- It involves quadratic structure, meaning its Laplace transform will involve convolution operations.
- The inner sum contains exponential decay terms e^{-t} and e^{-3t}, scaled by polynomial factors.
- The square indicates that the transform will involve convolutions or the need to compute the square of the transform of the inner sum.

Step-by-Step Process to Calculate $L\{F(t)\}$

Step 1: Recognize the structure of $F(t)$

Since $F(t)$ is a square of a sum, it can be written as:

$$
F(t) = \left(g(t) \right)^2, \quad \text{where} \quad g(t) = 1 + 3t e^{-t} + 4t e^{-3t}
$$

The Laplace transform of a square of a function is related to the convolution of the transform of the function with itself:

$$
L\{f(t)^2\} = L\{g(t)^2\} = \left(L\{g(t)\} \right) \left(L\{g(t)\} \right)
$$

where \(\) denotes convolution in the s-domain.

Step 2: Compute $L\{g(t)\}$

Break down $g(t)$ into simpler parts:

$$
g(t) = 1 + 3t e^{-t} + 4t e^{-3t}
$$

Apply linearity:

$$
L\{g(t)\} = L\{1\} + 3 L\{t e^{-t}\} + 4 L\{t e^{-3t}\}
$$

Now, compute each term separately.

Calculating the Laplace Transform of Each Term in $g(t)$

1. $L\{1\}$

The Laplace transform of a constant:

$$
L\{1\} = \frac{1}{s}
$$

2. $L\{t e^{-t}\}$

Recall the standard Laplace transform:

$$
L\{t e^{a t}\} = \frac{1}{(s - a)^2}
$$

for $\operatorname{Re}(s) > \operatorname{Re}(a)$.

Here, $a = -1$, so:

$$
L\{t e^{-t}\} = \frac{1}{(s + 1)^2}
$$

3. $L\{t e^{-3t}\}$

Similarly, with $a = -3$:

$$
L\{t e^{-3t}\} = \frac{1}{(s + 3)^2}
$$

Combining these results:

$$
L\{g(t)\} = \frac{1}{s} + 3 \cdot \frac{1}{(s + 1)^2} + 4 \cdot \frac{1}{(s + 3)^2}
$$

Step 3: Express the Laplace transform of $F(t)$

Since $F(t) = (g(t))^2$, its Laplace transform is the convolution of $L\{g(t)\}$ with itself:

$$
L\{F(t)\} = (L\{g(t)\}) (L\{g(t)\})
$$

which is:

\[

L\{F(t)\} = \int_0^{\infty} L\{g(\tau)\} \cdot L\{g(t - \tau)\} \, d \tau
\]

or, in the frequency domain:

\[
L\{F(t)\} = \left(L\{g(t)\} \right)^2
\]

Note: The convolution theorem states:

\[
L\{f(t) h(t)\} = L\{f(t)\} \cdot L\{h(t)\}
\]

and for the square, it becomes:

\[
L\{g(t)^2\} = \left(L\{g(t)\} \right)^2
\]

Step 4: Square \(L\{g(t)\}\)

Expressed explicitly:

\[
L\{g(t)\} = \frac{1}{s} + \frac{3}{(s + 1)^2} + \frac{4}{(s + 3)^2}
\]

Thus,

\[
L\{F(t)\} = \left(\frac{1}{s} + \frac{3}{(s + 1)^2} + \frac{4}{(s + 3)^2}
\right)^2
\]

This expression can be expanded to a sum of terms involving products of these fractions.

Step 5: Expand the Square

Applying the binomial expansion:

\[
L\{F(t)\} = \left(A + B + C \right)^2 = A^2 + B^2 + C^2 + 2AB + 2AC + 2BC
\]

where

$$
A = \frac{1}{s}, \quad B = \frac{3}{(s + 1)^2}, \quad C = \frac{4}{(s + 3)^2}
$$

Compute each term:

- $A^2 = \frac{1}{s^2}$
- $B^2 = \frac{9}{(s + 1)^4}$
- $C^2 = \frac{16}{(s + 3)^4}$
- $2AB = 2 \times \frac{1}{s} \times \frac{3}{(s + 1)^2} = \frac{6}{s (s + 1)^2}$
- $2AC = 2 \times \frac{1}{s} \times \frac{4}{(s + 3)^2} = \frac{8}{s (s + 3)^2}$
- $2BC = 2 \times \frac{3}{(s + 1)^2} \times \frac{4}{(s + 3)^2} = \frac{24}{(s + 1)^2 (s + 3)^2}$

Final Expression for the Laplace Transform

Putting it all together:

$$
\boxed{
L\{f(t)\} = \frac{1}{s^2} + \frac{9}{(s + 1)^4} + \frac{16}{(s + 3)^4} + \frac{6}{s (s + 1)^2} + \frac{8}{s (s + 3)^2} + \frac{24}{(s + 1)^2 (s + 3)^2}
}
$$

This is the comprehensive Laplace transform of $F(t)$.

Inverse Laplace Transform and Physical Interpretation

Once the transform is established, inverse Laplace techniques can be employed to find $f(t)$, or to analyze the behavior of the function in the s-domain, which is useful for system stability, response analysis, and solving differential equations.

Techniques for inverse transform:

- Partial fraction decomposition
- Recognizing standard Laplace pairs
- Convolution theorem for complex terms
- Numerical inversion methods for complicated expressions

Summary and Key Takeaways

- The function $F(t) = (1 + 3 t e^{-t} + 4 t e^{-3t})^2$ involves polynomial and exponential terms.
- To compute its Laplace transform, leverage the properties of linearity, and the convolution theorem.
- The Laplace transform of the inner sum $g(t)$ simplifies to a sum of rational functions

Frequently Asked Questions

How do I find the Laplace Transform L{F(t)} for the function F(t) = (1 + 3t e^{-t} + 4t e^{-3t})^2?

To find L{F(t)}, first expand the squared expression, then use the Laplace Transform linearity and known transforms for t, e^{-at} terms. Alternatively, consider simplifying the expression before applying the transform or using convolution properties if applicable.

What is the step-by-step process to compute the Laplace Transform of the squared function F(t) = (1 + 3t e^{-t} + 4t e^{-3t})^2?

First, expand the square to obtain a sum of terms. Next, find the Laplace Transform of each term, using standard transforms such as L{t e^{-at}} = 1/(s + a)^2. Then, sum the transforms to get the final L{F(t)}.

Can the Laplace Transform of a squared function be simplified using convolution, and how does it apply here?

Yes, the Laplace Transform of a squared function can be expressed as the convolution of the function with itself: L{f(t)^2} = (1/2πi) ∫_{γ - i∞}^{γ + i∞} F(s)^2 ds. In this case, since F(t) is a sum, you may also consider the convolution of the transforms of the individual components for simplification.

What are common pitfalls when calculating the Laplace Transform of a complex squared function like F(t) = (1 + 3t e^{-t} + 4t e^{-3t})^2?

Common pitfalls include incorrect expansion of the square, neglecting the linearity of the Laplace Transform, misapplying the transforms of exponential and polynomial terms, and overlooking the need to simplify or decompose the expression before transformation.

After computing the Laplace Transform L{F(t)}, how can I interpret or use the result in solving differential equations?

The Laplace Transform converts differential equations into algebraic equations in the s-domain. Once you find L{F(t)}, you can solve these algebraic equations more easily, then apply the inverse Laplace Transform to find solutions in the time domain.

Additional Resources

Calculating the Laplace Transform of a Complex Function: An Analytical Examination of F(t) = (1 + 3t e^{-t} + 4t e^{-3t})^2 and Its Implications

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Introduction

In the realm of mathematical analysis, the Laplace transform stands as a pivotal tool for transforming differential equations into algebraic ones, simplifying the process of solving complex systems across engineering, physics, and applied mathematics. Its utility extends beyond mere transformation; it offers deep insights into the behavior of functions, especially those involving exponential and polynomial terms. This article embarks on a comprehensive exploration of the Laplace transform applied to a particularly intricate function:

$$
F(t) = \left(1 + 3t e^{-t} + 4t e^{-3t}\right)^2
$$

Our goal is not only to compute $\mathcal{L}\{F(t)\}$ but also to analyze the methodology, interpret the results, and understand their broader significance in mathematical modeling and problem-solving contexts.

Overview of the Laplace Transform

Definition and Basic Properties

The Laplace transform of a function $f(t)$, defined for $t \geq 0$, is given by:

$$
\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t) \, dt
$$

This integral transform converts functions from the time domain into the complex frequency domain, enabling algebraic manipulation of differential equations. The primary advantages include:

- Simplification of differential equations into algebraic equations.
- Facility in handling initial conditions.
- Enhanced analysis of system stability and response.

Common Laplace Transform Pairs

Before delving into the specific function, it is beneficial to recall some standard transforms:

Function $f(t)$	Laplace Transform $\mathcal{L}\{f(t)\}$
1	$1/s$
t^n	$n! / s^{n+1}$
e^{at}	$1 / (s - a)$
$t e^{at}$	$1 / (s - a)^2$
$e^{at} \sin(bt)$	$b / [(s - a)^2 + b^2]$

These foundational pairs serve as building blocks in tackling more complex functions.

Decomposing the Function $F(t)$

Understanding the Structure of $F(t)$

The given function:

$$
F(t) = \left(1 + 3t e^{-t} + 4t e^{-3t}\right)^2
$$

is a square of a sum involving exponential and polynomial terms. To facilitate the Laplace transform computation, it is prudent to expand this square:

$$
F(t) = \left(1\right)^2 + 2 \times 1 \times (3t e^{-t} + 4t e^{-3t}) + (3t e^{-t} + 4t e^{-3t})^2
$$

which simplifies to:

$$
F(t) = 1 + 2(3t e^{-t} + 4t e^{-3t}) + (3t e^{-t} + 4t e^{-3t})^2
$$

The expansion yields three parts:

1. A constant term: 1,
2. A linear combination involving $t e^{-t}$ and $t e^{-3t}$,
3. The square of the sum $3t e^{-t} + 4t e^{-3t}$.

Addressing each component separately simplifies the overall process.

Step-by-Step Calculation of the Laplace Transform

1. Laplace Transform of the Constant Term

The simplest component is:

$$
\mathcal{L}\{1\} = \frac{1}{s}
$$

This is a standard result, serving as the baseline for further calculations.

2. Laplace Transform of $2 \times 3t e^{-t}$ and $2 \times 4t e^{-3t}$

Breaking down the second term:

$$
2 \times 3t e^{-t} = 6t e^{-t}
$$
$$
2 \times 4t e^{-3t} = 8t e^{-3t}
$$

Using the linearity of the Laplace transform:

$$
\mathcal{L}\{6t e^{-t}\} = 6 \times \mathcal{L}\{t e^{-t}\}
$$
$$
\mathcal{L}\{8t e^{-3t}\} = 8 \times \mathcal{L}\{t e^{-3t}\}
$$

Recall that:

$$
\mathcal{L}\{t e^{a t}\} = \frac{1}{(s - a)^2}
$$

for $s > a$. Therefore:

$$
\mathcal{L}\{t e^{-t}\} = \frac{1}{(s + 1)^2}
$$

$$
\mathcal{L}\{t e^{-3t}\} = \frac{1}{(s + 3)^2}
$$

Hence:

$$
\mathcal{L}\{6t e^{-t}\} = 6 \times \frac{1}{(s + 1)^2} = \frac{6}{(s + 1)^2}
$$

$$
\mathcal{L}\{8t e^{-3t}\} = 8 \times \frac{1}{(s + 3)^2} = \frac{8}{(s + 3)^2}
$$

Adding these results:

$$
\mathcal{L}\{2(3t e^{-t} + 4t e^{-3t})\} = \frac{6}{(s + 1)^2} + \frac{8}{(s + 3)^2}
$$

---

3. Laplace Transform of $(3t e^{-t} + 4t e^{-3t})^2$

This term involves the square of the sum:

$$
(3t e^{-t} + 4t e^{-3t})^2 = 9 t^2 e^{-2t} + 24 t^2 e^{-4t} + 2 \times 3t e^{-t} \times 4t e^{-3t}
$$

which expands to:

$$
9 t^2 e^{-2t} + 24 t^2 e^{-4t} + 24 t^2 e^{-(1 + 3)t} = 9 t^2 e^{-2t} + 24 t^2 e^{-4t} + 24 t^2 e^{-4t}
$$

But note the cross term:

$$
2 \times 3t e^{-t} \times 4t e^{-3t} = 24 t^2 e^{-(t + 3t)} = 24 t^2 e^{-4t}
$$

Adding this to the previous terms:

$$
(3t e^{-t} + 4t e^{-3t})^2 = 9 t^2 e^{-2t} + 48 t^2 e^{-4t}
$$

Calculating the Laplace Transform of the Quadratic Terms

The key is to compute the Laplace transform of:

$$
A t^2 e^{a t}
$$

which is given by:

$$
\mathcal{L}\{t^n e^{a t}\} = \frac{n!}{(s - a)^{n+1}}
$$

for $n = 2$, this becomes:

$$
\mathcal{L}\{t^2 e^{a t}\} = \frac{2!}{(s - a)^3} = \frac{2}{(s - a)^3}
$$

Applying this to each term:

- For $9 t^2 e^{-2t}$:

$$
\mathcal{L}\{9 t^2 e^{-2t}\} = 9 \times \frac{2}{(s + 2)^3} = \frac{18}{(s + 2)^3}
$$

- For $48 t^2 e^{-4t}$:

$$
\mathcal{L}\{48 t^2 e^{-4t}\} = 48 \times \frac{2}{(s + 4)^3} = \frac{96}{(s + 4)^3}
$$

Adding these:

$$
\mathcal{L}\left[(3 t e^{-t} + 4 t e^{-3t})^2\right] = \frac{18}{(s + 2)^3} + \frac{96}{(s + 4)^3}
$$

Consolidated Expression for $\mathcal{L}\{F(t)\}$

Putting all components together:

```
\[
\boxed{
\mathcal{L}\{F(t)\} = \frac{1}{s} + \frac{6}{(s + 1
```

Calculate The Laplace Transform L F T For The Function F T 1 3te T 4te 3t 2 And Then Determine

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