Consider A Sequence Where A0 = 1, A1 = 2, And An = 2an1 An2 For N 2. Guess An As A Function Of N And

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Understanding recursive sequences can be both fascinating and challenging. In this article, we explore a particular sequence defined by initial conditions and a recursive formula, aiming to determine a closed-form expression for An as a function of N. Specifically, we analyze the sequence where A0 = 1, A1 = 2, and for N ≥ 2, An = 2 A(n−1) A(n−2). This problem involves recognizing the pattern, formulating conjectures, and applying mathematical techniques such as characteristic equations and induction to uncover the explicit formula for An.

Analyzing the Recursive Sequence

Understanding the Initial Conditions and Recursion

The sequence is given by:

- A0 = 1
- A1 = 2
- For N ≥ 2, An = 2 A(n−1) A(n−2)

This recursive relation suggests that each term depends multiplicatively on the previous two terms, scaled by a factor of 2. Such relations are common in sequences related to Fibonacci-like sequences but involve multiplicative rather than additive recurrences.

Computing the First Few Terms

To better understand the pattern, compute the initial terms:

- A0 = 1
- A1 = 2

- A2 = 2 A1 A0 = 2 2 1 = 4
- A3 = 2 A2 A1 = 2 4 2 = 16
- A4 = 2 A3 A2 = 2 16 4 = 128
- A5 = 2 A4 A3 = 2 128 16 = 4096

Observing these values:

- A0 = 1
- A1 = 2
- A2 = 4
- A3 = 16
- A4 = 128
- A5 = 4096

The sequence exhibits rapid growth, suggesting an exponential or super-exponential pattern.

Identifying the Pattern and Formulating a Guess

Recognizing the Pattern in the Sequence

Looking at the computed terms:

- A0 = 1
- A1 = 2
- A2 = 4 = 2^2
- A3 = 16 = 2^4
- A4 = 128 = 2^7
- A5 = 4096 = 2^12

Let's write these as powers of 2:

N	A_N	Power of 2
0	1	2^0
1	2	2^1
2	4	2^2
3	16	2^4
4	128	2^7
5	4096	2^12

Note the exponents:

- 0
- 1
- 2
- 4
- 7
- 12

Sequence of exponents: 0, 1, 2, 4, 7, 12

We observe that the exponents follow a pattern:

- 0 (N=0)
- 1 (N=1)
- 2 (N=2)
- 4 (N=3)
- 7 (N=4)
- 12 (N=5)

Looking at the differences:

- 1 - 0 = 1
- 2 - 1 = 1
- 4 - 2 = 2
- 7 - 4 = 3
- 12 - 7= 5

Differences between exponents: 1, 1, 2, 3, 5

These are Fibonacci-like differences, hinting that the exponents themselves might follow a recursive pattern similar to Fibonacci numbers.

Observation: The differences are Fibonacci numbers:

- 1, 1, 2, 3, 5

which are Fibonacci numbers starting from 1, 1, 2, 3, 5.

Therefore, the exponents could be modeled as sums of Fibonacci numbers, or as a sequence related to Fibonacci numbers.

Expressing the Exponents as a Function of N

Suppose the exponent E(N) satisfies:

$E(0) = 0$

and for $N \geq 1$,

$E(N) = E(N-1) + F(N-1)$

where $F(N-1)$ is the Fibonacci number at position $N-1$.

Check if this fits:

- For $N=0$: $E(0)=0$
- $N=1$: $E(1)=E(0)+F(0)=0+1=1$ (since $F(0)=1$)
- $N=2$: $E(2)=E(1)+F(1)=1+1=2$
- $N=3$: $E(3)=E(2)+F(2)=2+2=4$
- $N=4$: $E(4)=E(3)+F(3)=4+3=7$
- $N=5$: $E(5)=E(4)+F(4)=7+5=12$

This matches our observed exponents exactly.

Thus, we have:

$$E(N) = \sum_{k=0}^{N-1} F(k)$$

Since $\sum_{k=0}^{n} F(k) = F(n+2) - 1$, by the Fibonacci sum formula.

Therefore:

$$E(N) = \sum_{k=0}^{N-1} F(k) = F(N+1) - 1$$

Result:

$$A_N = 2^{E(N)} = 2^{F(N+1) - 1}$$

Deriving the Closed-Form Formula for A_N

Final Expression for the Sequence

Based on the pattern recognition and Fibonacci identities, the explicit formula for An is:

$$AN = 2^{F(N+1) - 1}$$

where $F(N+1)$ is the $(N+1)$-th Fibonacci number, with $F(0)=0$, $F(1)=1$.

Fibonacci Numbers and Their Closed-Form (Binet's Formula)

The Fibonacci numbers can be expressed explicitly using Binet's formula:

$F(N) = (\varphi^N - \psi^N) / \sqrt{5}$

where:

- $\varphi = (1 + \sqrt{5}) / 2 \approx 1.61803$ (the golden ratio)
- $\psi = (1 - \sqrt{5}) / 2 \approx -0.61803$

Thus,

$$AN = 2^{(F(N+1) - 1)} = 2^{((\varphi^{N+1} - \psi^{N+1}) / \sqrt{5}) - 1}$$

This provides a precise closed-form expression involving radicals.

Summary and Final Formula

To summarize, the sequence defined by:

- $A0 = 1$
- $A1 = 2$
- For $N \geq 2$, $An = 2 A(n-1) A(n-2)$

has the explicit formula:

Explicit Formula for An

$$An = 2^{F(N+1) - 1}$$

where $F(N+1)$ is the $(N+1)$-th Fibonacci number, with $F(0)=0$, $F(1)=1$.

Therefore, the sequence grows exponentially, with each term determined directly from Fibonacci numbers.

Implications and Applications

Understanding this sequence has several implications, particularly in combinatorics, algorithm analysis, and mathematical modeling where recursive multiplicative relations appear. Recognizing Fibonacci patterns in sequences helps simplify complex recursive definitions and find closed-form solutions, making calculations more straightforward and insightful.

Conclusion

By carefully analyzing initial terms, recognizing Fibonacci-like patterns, and applying known identities, we've deduced that the sequence An can be expressed explicitly as a power of 2 involving Fibonacci numbers. This approach exemplifies how pattern recognition and mathematical identities can unlock the secrets behind recursive sequences, transforming complex recursions into simple, closed-form formulas suitable for advanced analysis and application.

Frequently Asked Questions

What is the initial sequence given in the problem?

The sequence starts with $A_0 = 1$ and $A_1 = 2$, and for $N \geq 2$, $A_n = 2 A(n-1) A(n-2)$.

How can we interpret the recurrence relation An = 2 A(n-1) A(n-2)?

It indicates that each term is twice the product of the previous two terms, suggesting a multiplicative and exponential pattern.

What is a common approach to find a closed-form expression for such sequences?

Using logarithms to convert the recurrence into a linear form, then solving the resulting linear recurrence to find an explicit formula.

How can we transform the recurrence relation to simplify solving for An?

Define $L_n = \log(A_n)$, which turns the recurrence into $L_n = \log(2) + L_{n-1} + L_{n-2}$, a linear recurrence in terms of L_n.

What is the general solution to the linear recurrence Ln = log(2) + Ln-1 + Ln-2?

The homogeneous part has characteristic roots, leading to a solution of the form $L_n = A r_1^n + B r_2^n$ + particular solution, where r_1 and r_2 are roots of the characteristic equation.

Based on the recurrence and initial conditions, what is the guessed closed-form for An?

$A_n = 2^{(n(n-1))/2}$, which can be derived by solving the recurrence in logarithmic form and exponentiating back.

Additional Resources

Exploring the Sequence: A Deep Dive into the Recurrence Relation $A_n = 2A_{n-1}A_{n-2}$

Understanding sequences defined by recurrence relations is foundational in mathematical analysis, combinatorics, and computer science. The sequence in question, where $A_0 = 1$, $A_1 = 2$, and $A_n = 2A_{n-1}A_{n-2}$ for $N \geq 2$, offers rich ground for exploration—particularly in discovering a closed-form expression for A_n as a function of N. This review aims to dissect this sequence thoroughly, examining its properties, deriving explicit formulas, and understanding its behavior.

Introduction to the Sequence

The sequence is defined recursively with initial conditions:

- $A_0 = 1$
- $A_1 = 2$
- For $N \geq 2$, $A_n = 2A_{n-1}A_{n-2}$

At first glance, this recurrence resembles multiplicative sequences, differing from the more common additive recursions like Fibonacci or Lucas sequences. Its multiplicative nature suggests that the sequence's growth might be exponential or super-exponential, and that the sequence could be expressed in terms of products or factorials.

Initial Terms and Pattern Recognition

Calculating the first few terms provides insight:

- $A_0 = 1$
- $A_1 = 2$
- $A_2 = 2 A_1 A_0 = 2 2 1 = 4$
- $A_3 = 2 A_2 A_1 = 2 4 2 = 16$
- $A_4 = 2 A_3 A_2 = 2 16 4 = 128$
- $A_5 = 2 A_4 A_3 = 2 128 16 = 4096$

Sequence terms:

N	A_n
0	1
1	2
2	4

3	16
4	128
5	4096

Plotting these points indicates rapid growth, hinting at exponential or perhaps double-exponential behavior.

Deriving a General Pattern

To find a closed-form formula, analyze the recurrence:

$A_n = 2A_{n-1}A_{n-2}$

Taking natural logarithms to simplify the multiplicative recurrence:

Let $B_n = \ln(A_n)$. Then:

$B_n = \ln(2) + \ln(A_{n-1}) + \ln(A_{n-2}) = \ln(2) + B_{n-1} + B_{n-2}$

This transforms the original recurrence into:

$B_n = B_{n-1} + B_{n-2} + \ln(2)$

Solving the Logarithmic Recurrence

The recurrence for B_n:

$B_n - B_{n-1} - B_{n-2} = \ln(2)$

is a non-homogeneous linear recurrence relation.

Step 1: Homogeneous Solution

Consider the homogeneous recurrence:

$B_n^h - B_{n-1}^h - B_{n-2}^h = 0$

Characteristic equation:

$r^2 - r - 1 = 0$

Solving:

$r = [1 \pm \sqrt{(1 + 4)}]/2 = [1 \pm \sqrt{5}]/2$

Define:

$r_1 = (1 + \sqrt{5})/2$ (the golden ratio, φ)

$r_2 = (1 - \sqrt{5})/2 \approx -0.618...$

Homogeneous solution:

$B_n\hat{\ }h = \alpha r_1^n + \beta r_2^n$

where α, β are constants determined by initial conditions.

Step 2: Particular Solution

Since the non-homogeneous term is a constant ln(2), assume a particular solution:

$B_n\hat{\ }p = C$ (a constant)

Plug into the recurrence:

$C - C - C = \ln(2)$

which simplifies to:

$0 = \ln(2)$

Contradiction unless the particular solution is not constant.

Instead, try a linear particular solution:

$B_n\hat{\ }p = D n$

Plug into the recurrence:

$D n - D (n - 1) - D (n - 2) = \ln(2)$

Compute:

$D n - D n + D - D n + 2 D = \ln(2)$

Simplify:

$(D) + (2 D) = \ln(2) \rightarrow 3 D = \ln(2) \rightarrow D = (1/3) \ln(2)$

Thus:

$B_n\hat{\ }p = (1/3) \ln(2) n$

General Solution for B_n

Combining homogeneous and particular solutions:

$B_n = \alpha r_1^n + \beta r_2^n + (\ln(2)/3) n$

Using initial conditions:

- For n=0: $A_0=1 \rightarrow B_0=0$

- For n=1: $A_1=2 \rightarrow B_1=\ln(2)$

Determining Constants α and β

Calculate:

$B_0 = \alpha r_1^0 + \beta r_2^0 + 0 = \alpha + \beta = 0$

$\rightarrow \beta = -\alpha$

Next,

$B_1 = \alpha r_1 + \beta r_2 + (\ln(2)/3) 1$

Substitute $\beta = -\alpha$:

$B_1 = \alpha r_1 - \alpha r_2 + (\ln(2)/3)$

Set equal to ln(2):

$\alpha (r_1 - r_2) + (\ln(2)/3) = \ln(2)$

Recall:

$r_1 - r_2 = \sqrt{5}$

Therefore:

$\alpha \sqrt{5} + (\ln(2)/3) = \ln(2)$

Solve for α:

$\alpha \sqrt{5} = \ln(2) - (\ln(2)/3) = (2/3) \ln(2)$

$\rightarrow \alpha = [(2/3) \ln(2)] / \sqrt{5}$

Similarly,

$\beta = -\alpha = - [(2/3) \ln(2)] / \sqrt{5}$

Explicit Expression for B_n

Now,

$B_n = \alpha r_1^n + \beta r_2^n + (\ln(2)/3) n$

Substitute α and β:

$B_n = [(2/3) \ln(2)/\sqrt{5}] r_1^n - [(2/3) \ln(2)/\sqrt{5}] r_2^n + (\ln(2)/3) n$

Factor out common terms:

$B_n = (2/3) \ln(2)/\sqrt{5} (r_1^n - r_2^n) + (\ln(2)/3) n$

Recall $r_1 = (1 + \sqrt{5})/2$, $r_2 = (1 - \sqrt{5})/2$

Expressing A_n as a Closed-Form Formula

Since $A_n = e^{B_n}$, then:

$A_n = \exp[(2/3) \ln(2)/\sqrt{5} (r_1^n - r_2^n) + (\ln(2)/3) n]$

Break into product:

$A_n = \exp[(2/3) \ln(2)/\sqrt{5} (r_1^n - r_2^n)] \exp[(\ln(2)/3) n]$

Note that:

$\exp[(\ln(2)/3) n] = 2^{n/3}$

and

$\exp[(2/3) \ln(2)/\sqrt{5} (r_1^n - r_2^n)] = 2^{(2/3)/\ln(2) (\ln(2)/\sqrt{5}) (r_1^n - r_2^n)} = 2^{(2/3)/\ln(2) (\ln(2)/\sqrt{5}) (r_1^n - r_2^n)}$ (which simplifies back to an exponential in $r_1^n - r_2^n$)

Alternatively, to keep it straightforward:

Final explicit formula:

$$
A_n = 2^{n/3} \times \exp \left[\frac{2}{3} \times \frac{\ln(2)}{\sqrt{5}} \times (r_1^n - r_2^n) \right]
$$

where

$$
r_1 = \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad r_2 = \frac{1 - \sqrt{5}}{2}
$$

Asymptotic Behavior and Growth Rate

Examining the dominant term:

- Since $|r_1| > 1$ and $|r_2| < 1$, for large n, r_1^n dominates r_2^n.
- The sequence A_n grows approximately like:

$$
A_n \sim 2^{n/3} \times \exp \left[\frac
$$

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