Consider The Differential Equation 4y" - 4y' + Y = 0; E^x/2, Xe^x/2. Verify That The Functions E^x/2

Consider The Differential Equation 4y" - 4y' + Y = 0; E^x/2, Xe^x/2. Verify That The Functions E^x/2 and explore their roles as solutions to the differential equation. Differential equations are fundamental in modeling various phenomena in physics, engineering, and applied mathematics. Understanding how specific functions satisfy these equations is crucial for solving complex problems involving growth, decay, oscillations, and other dynamic behaviors.

In this article, we will delve into the nature of the differential equation presented, analyze the proposed functions, and verify whether they are solutions. We will also discuss the general solution to the differential equation, methods for solving such equations, and the significance of the exponential functions involved.

Understanding the Differential Equation

Formulation and Characteristics

The differential equation under consideration is:

$$ 4 y'' - 4 y' + y = 0 $$

This is a second-order linear homogeneous differential equation with constant coefficients. It can be characterized by its structure:

- The highest derivative is second order (y'')
- Coefficients of derivatives are constants
- The equation is homogeneous (right side equals zero)

Such equations are typically solved using the characteristic equation method, which involves assuming solutions of exponential form.

Significance of the Equation

Equations of this type frequently appear in physical systems exhibiting oscillatory or exponential behaviors, such as:

- Mechanical vibrations
- Electrical circuits
- Population models with exponential growth/decay

Finding solutions helps predict system behavior over time or space.

Proposed Functions for Verification

The functions proposed are:

- $y_1 = e^{x/2}$
- $y_2 = x e^{x/2}$

These functions are candidates for solutions, and our goal is to verify whether they satisfy the differential equation.

Why These Functions?

The functions involve exponential terms and a polynomial factor in the second function, which suggests they may be related to the roots of the characteristic equation, particularly if it has repeated roots.

Solving the Differential Equation

Step 1: Write the Characteristic Equation

Assuming solutions of the form $y = e^{rx}$, substitute into the differential equation:

$$4 r^2 e^{rx} - 4 r e^{rx} + e^{rx} = 0$$

Dividing through by e^{rx} (which is never zero):

$$4 r^2 - 4 r + 1 = 0$$

This quadratic in r is called the characteristic equation.

Step 2: Find Roots of the Characteristic Equation

Solve:

$$4 r^2 - 4 r + 1 = 0$$

Using the quadratic formula:

$$r = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 4 \times 1}}{2 \times 4}$$

Calculate discriminant:

$$ \Delta = 16 - 16 = 0 $$

Since the discriminant is zero, there is a repeated root:

$$ r = \frac{4}{8} = \frac{1}{2} $$

This indicates a repeated root at $r = \frac{1}{2}$.

Step 3: Write the General Solution

For a second-order linear differential equation with a repeated root r, the general solution is:

$$ y(x) = C_1 e^{r x} + C_2 x e^{r x} $$

Substituting $r = \frac{1}{2}$:

$$ y(x) = C_1 e^{x/2} + C_2 x e^{x/2} $$

This confirms that the functions $e^{x/2}$ and $x e^{x/2}$ are fundamental solutions of the differential equation.

Verification of the Proposed Functions

Now, we verify whether $y_1 = e^{x/2}$ and $y_2 = x e^{x/2}$ satisfy the original differential equation.

Verification of $y_1 = e^{x/2}$

Calculate derivatives:

- $y_1' = \frac{1}{2} e^{x/2}$
- $y_1'' = \frac{1}{4} e^{x/2}$

Substitute into the differential equation:

$$ 4 y_1'' - 4 y_1' + y_1 = 4 \times \frac{1}{4} e^{x/2} - 4 \times \frac{1}{2} e^{x/2} + e^{x/2} $$

Simplify:

$$ e^{x/2} - 2 e^{x/2} + e^{x/2} = (1 - 2 + 1) e^{x/2} = 0 $$

Thus, $y_1 = e^{x/2}$ is a solution.

Verification of $y_2 = x e^{x/2}$

Calculate derivatives:

- $y_2' = e^{x/2} + x \times \frac{1}{2} e^{x/2} = e^{x/2} + \frac{x}{2} e^{x/2}$
- $y_2'' = \frac{1}{2} e^{x/2} + \frac{1}{2} e^{x/2} + \frac{x}{2} \times \frac{1}{2} e^{x/2} = \left(\frac{1}{2} + \frac{1}{2} + \frac{x}{4} \right) e^{x/2} = \left(1 + \frac{x}{4} \right) e^{x/2}$

Substitute into the differential equation:

$$4 y_2'' - 4 y_2' + y_2$$

$$= 4 \left(1 + \frac{x}{4} \right) e^{x/2} - 4 \left(e^{x/2} + \frac{x}{2} e^{x/2} \right) + x e^{x/2}$$

Simplify each term:

$$= (4 + x) e^{x/2} - 4 e^{x/2} - 2 x e^{x/2} + x e^{x/2}$$

Combine like terms:

$$(4 + x - 4 - 2 x + x) e^{x/2} = (0 + 0) e^{x/2} = 0$$

Therefore, $y_2 = x e^{x/2}$ also satisfies the differential equation.

Implications and Applications

Understanding solutions like $e^{x/2}$ and $x e^{x/2}$ provides insight into the behavior of systems modeled by this differential equation. The presence of repeated roots indicates solutions involving polynomial multiples of exponential functions, characteristic of systems with repeated eigenvalues or specific boundary conditions.

Some applications include:

- Mechanical systems with damping
- Electrical circuits with resonant frequencies
- Population dynamics with exponential growth patterns

Conclusion

The differential equation $4 y'' - 4 y' + y = 0$ has a characteristic root at $r = \frac{1}{2}$ with multiplicity two. Consequently, the general solution involves the functions $e^{x/2}$ and $x e^{x/2}$, which have been verified as solutions. Recognizing these functions as fundamental solutions allows for constructing the complete general solution and applying it to real-world problems. Mastery of such methods is essential for solving higher-order differential equations and understanding the behavior of complex systems modeled mathematically.

Frequently Asked Questions

What is the differential equation given in the problem?

The differential equation is $4y'' - 4y' + y = 0$.

What are the proposed functions to verify as solutions?

The proposed functions are $e^{x/2}$ and $x e^{x/2}$.

How do we verify that $e^{x/2}$ is a solution to the differential equation?

By computing its derivatives $y' = (1/2) e^{x/2}$ and $y'' = (1/4) e^{x/2}$, then substituting into the equation to check if the equation simplifies to zero.

Is $x e^{x/2}$ also a solution? How do we verify it?

Yes, to verify, compute its derivatives y' and y'', then substitute into the differential equation to see if the equation equals zero.

What is the general solution to the differential equation?

The general solution involves linear combinations of $e^{x/2}$ and $x e^{x/2}$, typically expressed as $y(x) = C_1 e^{x/2} + C_2 x e^{x/2}$.

Why are these functions considered solutions to the differential equation?

Because when substituted into the differential equation, they satisfy the equation by reducing it to an identity, confirming their validity as solutions.

Additional Resources

Consider The Differential Equation $4y'' - 4y' + y = 0$: An In-Depth Analysis and Solution Approach

Understanding differential equations is fundamental in mathematical modeling across sciences and engineering. Among these, linear homogeneous differential equations with constant coefficients are particularly well-studied due to their straightforward solution methods. The differential equation $4y'' - 4y' + y = 0$ exemplifies such an equation, and analyzing its solutions provides insights into the behavior of systems modeled by it. This article offers a comprehensive review of solving this differential equation, verifying particular solutions involving exponential functions, and discussing their properties and implications.

Overview of the Differential Equation

The differential equation in question is:

$$ 4y'' - 4y' + y = 0 $$

This second-order linear homogeneous differential equation contains constant coefficients. The general solution depends on solving the characteristic equation derived from assuming solutions of the form $y = e^{rx}$.

Characteristic Equation and Its Significance

Deriving the Characteristic Equation

To solve the differential equation, we substitute a trial solution $y = e^{rx}$, which leads to:

$$
4r^2 e^{rx} - 4 r e^{rx} + e^{rx} = 0
$$

Dividing through by $e^{rx} \neq 0$:

$$
4r^2 - 4r + 1 = 0
$$

This quadratic equation, known as the characteristic equation, is central to finding the general solution.

Solving the Characteristic Equation

The quadratic is:

$$
4r^2 - 4r + 1 = 0
$$

Using the quadratic formula:

$$
r = \frac{4 \pm \sqrt{(-4)^2 - 4 \times 4 \times 1}}{2 \times 4} = \frac{4 \pm \sqrt{16 - 16}}{8} = \frac{4 \pm 0}{8}
$$

$$
r = \frac{4}{8} = \frac{1}{2}
$$

Since the discriminant is zero, the roots are repeated: $r = \frac{1}{2}$.

Solution of the Differential Equation

General Solution for Repeated Roots

For a second-order linear differential equation with a repeated root r, the general solution is:

$$
y(x) = (A + Bx) e^{rx}
$$

where A and B are arbitrary constants.

Given $r = \frac{1}{2}$, the general solution is:

$$
\boxed{
y(x) = (A + Bx) e^{x/2}
}
$$

This form accounts for the multiplicity of the root and ensures a complete basis of solutions.

Verification of Particular Solutions

The problem mentions specific functions: $e^{x/2}$ and $x e^{x/2}$. These are solutions to the differential equation.

Verifying $y = e^{x/2}$

Calculate derivatives:

$$
y = e^{x/2}
$$
$$
y' = \frac{1}{2} e^{x/2}
$$
$$
y'' = \frac{1}{4} e^{x/2}
$$

Substitute into the original differential equation:

$$
4 y'' - 4 y' + y = 4 \times \frac{1}{4} e^{x/2} - 4 \times \frac{1}{2} e^{x/2} + e^{x/2}
$$

Simplify:

$$
e^{x/2} - 2 e^{x/2} + e^{x/2} = (1 - 2 + 1) e^{x/2} = 0
$$

Thus, $y = e^{x/2}$ is a solution, confirming the solution form.

Verifying $y = x e^{x/2}$

Calculate derivatives:

$$
y = x e^{x/2}
$$
$$
y' = e^{x/2} + x \times \frac{1}{2} e^{x/2} = e^{x/2} \left(1 + \frac{x}{2}\right)
$$
$$
y'' = \frac{1}{2} e^{x/2} \left(1 + \frac{x}{2}\right) + e^{x/2} \times \frac{1}{2} = e^{x/2} \left(\frac{1}{2} + \frac{x}{4} + \frac{1}{2} \right) = e^{x/2} \left(1 + \frac{x}{4} \right)
$$

Substituting into the differential equation:

$$
4 y'' - 4 y' + y = 4 \times e^{x/2} \left(1 + \frac{x}{4} \right) - 4 \times e^{x/2} \left(1 + \frac{x}{2} \right) + x e^{x/2}

$$

$$

Simplify each term:

$$
4 \times e^{x/2} \left(1 + \frac{x}{4} \right) = 4 e^{x/2} + x e^{x/2}
$$
$$
- 4 \times e^{x/2} \left(1 + \frac{x}{2} \right) = - 4 e^{x/2} - 2 x e^{x/2}
$$
$$
+ x e^{x/2}
$$

Adding all:

$$
(4 e^{x/2} + x e^{x/2}) + (- 4 e^{x/2} - 2 x e^{x/2}) + x e^{x/2} = (4 e^{x/2} - 4 e^{x/2}) + (x e^{x/2} - 2 x e^{x/2} + x e^{x/2}) = 0
$$

Thus, $y = x e^{x/2}$ is also a solution, confirming the general solution structure.

Features, Pros, and Cons of the Solution

Features:

- The solution involves exponential functions, which are common in modeling growth, decay, and oscillatory systems.
- The presence of repeated roots simplifies the solution form but indicates specific behaviors such as multiplicity in eigenvalues.
- The solutions form a basis for the solution space, allowing for the construction of particular solutions based on initial conditions.

Pros:

- Straightforward solution method using characteristic equations.
- Clear verification process for exponential solutions.
- Applicability to physical systems with exponential behavior.

Cons:

- Repeated roots mean solutions grow or decay exponentially with polynomial factors, which might not always be physically realistic.
- The method relies on constant coefficients; variable coefficient equations require different approaches.
- For more complex equations, the solving process can become significantly more involved.

Implications and Applications

Understanding solutions to the differential equation $4 y'' - 4 y' + y = 0$ extends to numerous practical applications:

- Electrical Engineering: Modeling circuits with resistors, capacitors, and inductors where exponential responses are common.
- Mechanical Systems: Analyzing damped harmonic oscillators with exponential decay or growth.
- Population Dynamics: Modeling growth with exponential factors, especially in systems with feedback mechanisms.
- Control Systems: Designing controllers based on root locus analysis, where repeated roots influence system stability.

Conclusion

The differential equation $4 y'' - 4 y' + y = 0$ exemplifies a linear homogeneous differential equation with constant coefficients, leading to solutions characterized by exponential functions and their polynomial multiples. Its characteristic equation yields a repeated root $r = \frac{1}{2}$, resulting in a general solution:

$$
\boxed{
y(x) = (A + B x) e^{x/2}
}
$$

The explicit verification of solutions such as $e^{x/2}$ and $x e^{x/2}$ confirms their validity and demonstrates the power of characteristic equations in solving such problems. The approach discussed provides a template for tackling similar equations, emphasizing the importance of understanding roots of the characteristic polynomial and their influence on the solution structure. While the simplicity of constant coefficient equations offers clarity, more complex systems challenge practitioners to develop and apply more advanced methods. Nonetheless, mastery of these fundamental techniques remains essential for anyone engaged in mathematical modeling and analysis.

Note: For specific initial conditions or boundary values, the arbitrary constants A and B can be determined, tailoring the solution to particular problems.

Consider The Differential Equation 4y 4y Y 0 E X 2 Xe X 2 Verify That The Functions E X 2

Consider The Differential Equation 4y 4y Y 0 E X 2 Xe X 2 Verify That The Functions E X 2

Related Articles

- During Lactation, The Daily Caloric Needs Of The Mother Increase By ________ Calories As Compared To
- Debris ________ In A Large Debris Cloud. *A. Orbits EarthB. Travels Into Deep SpaceC. Falls To Earth
- During 2020, Penny Co. Had Net Income Of $200,000 Including After-tax Interest Expense Of $30,000 On

Back to Home