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Consider The Differential Equation Y" - 6y' + 9y= 0.(a) Verify That Y = E^{3x} And Y2 = Xe^{3x} Are

Understanding and solving differential equations is a fundamental aspect of advanced mathematics, especially in fields like physics, engineering, and applied sciences. One such differential equation that often appears in coursework and research is the second-order linear homogeneous differential equation with constant coefficients:

$$Y'' - 6Y' + 9Y = 0$$

In this article, we will focus on verifying that the functions $Y = e^{3x}$ and $Y_2 = x e^{3x}$ are solutions to this differential equation. These functions are classical examples linked to the characteristic equation's roots and demonstrate important concepts such as repeated roots and the method of solving homogeneous differential equations.

Understanding the Differential Equation $Y'' - 6Y' + 9Y = 0$

Before diving into the verification process, it’s essential to understand the structure and characteristics of the differential equation.

Homogeneous Linear Differential Equations with Constant Coefficients

The differential equation in question is:

$$Y'' - 6Y' + 9Y = 0$$

which is a second-order linear homogeneous differential equation with constant coefficients. Such equations are typically solved by finding the roots of the characteristic (auxiliary) equation.

Characteristic Equation and Roots

The associated characteristic equation is obtained by replacing Y'' with r^2, Y' with r, and Y with 1:

$$ r^2 - 6r + 9 = 0 $$

Solving this quadratic provides the roots that determine the form of the general solution.

Solving the Characteristic Equation

Finding the Roots of the Characteristic Equation

The quadratic equation:

$$ r^2 - 6r + 9 = 0 $$

can be solved using factoring, quadratic formula, or completing the square.

Applying the quadratic formula:

$$ r = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 9}}{2 \times 1} $$
$$ r = \frac{6 \pm \sqrt{36 - 36}}{2} $$
$$ r = \frac{6 \pm 0}{2} $$
$$ r = 3 $$

The discriminant is zero, indicating a repeated root $r = 3$.

Implications of Repeated Roots

Since the characteristic equation has a repeated root, the general solution to the differential equation takes a specific form:

$$ y(x) = (A + Bx) e^{rx} $$

where A and B are arbitrary constants, and r is the repeated root (here, $r = 3$).

Verification of the Solutions $Y = e^{3x}$ and $Y_2 = x e^{3x}$

Now that we understand the form of the solutions, the next step is to verify that these functions satisfy the original differential equation.

1. Verifying $Y = e^{3x}$

Let's substitute $Y = e^{3x}$ into the differential equation:

$$Y'' - 6Y' + 9Y = 0$$

First, compute the derivatives:

- $Y' = \frac{d}{dx} e^{3x} = 3 e^{3x}$
- $Y'' = \frac{d}{dx} (3 e^{3x}) = 9 e^{3x}$

Substitute these into the differential equation:

$$9 e^{3x} - 6 \times 3 e^{3x} + 9 e^{3x} = 0$$

Simplify:

$$9 e^{3x} - 18 e^{3x} + 9 e^{3x} = (9 - 18 + 9) e^{3x} = 0$$

$$0 \times e^{3x} = 0$$

Thus, $Y = e^{3x}$ is a solution.

2. Verifying $Y_2 = x e^{3x}$

Next, verify $Y_2 = x e^{3x}$:

Compute derivatives:

- First derivative:

$$Y_2' = \frac{d}{dx} (x e^{3x}) = e^{3x} + x \times 3 e^{3x} = e^{3x} + 3x e^{3x}$$

- Second derivative:

$$Y_2'' = \frac{d}{dx} (e^{3x} + 3x e^{3x})$$

Calculate term by term:

$$ \frac{d}{dx} e^{3x} = 3 e^{3x} $$

$$ \frac{d}{dx} (3x e^{3x}) = 3 e^{3x} + 3x \times 3 e^{3x} = 3 e^{3x} + 9x e^{3x} $$

Add these:

$$ Y_2'' = 3 e^{3x} + 3 e^{3x} + 9x e^{3x} = 6 e^{3x} + 9x e^{3x} $$

Now, substitute Y_2, Y_2', and Y_2'' into the original equation:

$$ Y_2'' - 6 Y_2' + 9 Y_2 = 0 $$

Plugging in:

$$ (6 e^{3x} + 9x e^{3x}) - 6 (e^{3x} + 3x e^{3x}) + 9 x e^{3x} $$

Simplify term-by-term:

$$ 6 e^{3x} + 9x e^{3x} - 6 e^{3x} - 18 x e^{3x} + 9 x e^{3x} $$

Combine like terms:

- $6 e^{3x} - 6 e^{3x} = 0$
- $9x e^{3x} - 18 x e^{3x} + 9 x e^{3x} = (9x - 18x + 9x) e^{3x} = (0) e^{3x}$

Sum total:

$$ 0 + 0 = 0 $$

Hence, $Y_2 = x e^{3x}$ also satisfies the differential equation.

Conclusion: The Solutions Are Confirmed

Through direct substitution and differentiation, both functions $Y = e^{3x}$ and $Y_2 = x e^{3x}$ are verified solutions to the differential equation $Y'' - 6Y' + 9Y = 0$. These solutions align with the theory of second-order linear differential equations with repeated roots, where the general solution is:

$$ y(x) = (A + Bx) e^{3x} $$

Understanding these solutions is crucial for solving similar differential equations in various scientific disciplines, including physics, engineering, and mathematics. Recognizing the form of solutions based on the roots of the characteristic equation allows for efficient and accurate problem-solving in advanced differential equations.

Additional Resources for Solving Differential Equations

If you're interested in further exploring differential equations, consider the following topics:

- Methods for solving nonhomogeneous differential equations
- Laplace transforms in differential equations
- Applications of differential equations in physics and engineering
- Numerical methods for differential equations

Mastering these concepts enhances problem-solving skills and deepens understanding of mathematical modeling in real-world scenarios.

Keywords: differential equation solutions, homogeneous second-order equations, characteristic roots, repeated roots, exponential solutions, verifying solutions, calculus, differential equations tutorial

Frequently Asked Questions

What is the differential equation given in the problem?

The differential equation is $Y'' - 6Y' + 9Y = 0$.

What are the proposed solutions to verify for the differential equation?

The proposed solutions are $Y_1 = e^{3x}$ and $Y_2 = xe^{3x}$.

How do you verify that $Y_1 = e^{3x}$ is a solution?

Substitute Y_1 and its derivatives into the differential equation to check if it satisfies the equation.

What are the derivatives of $Y_1 = e^{3x}$?

$Y_1' = 3e^{3x}$ and $Y_1'' = 9e^{3x}$.

How is $Y_2 = xe^{3x}$ differentiated to verify the solution?

$Y_2' = e^{3x} + 3xe^{3x}$ and $Y_2'' = 2e^{3x} + 6xe^{3x}$.

What is the purpose of verifying these solutions?

To confirm that Y_1 and Y_2 satisfy the differential equation, indicating they are solutions.

What does the form of solutions suggest about the nature of the differential equation?

It suggests that the differential equation has repeated roots in its characteristic equation, leading to solutions involving e^{3x} and xe^{3x}.

How do you verify that $Y_2 = xe^{3x}$ is a solution?

Substitute Y_2 and its derivatives into the original differential equation to see if it equals zero.

What is the significance of these solutions in the context of differential equations?

They form the general solution to the differential equation, capturing all possible solutions due to its linearity and constant coefficients.

Additional Resources

Comprehensive Analysis of the Differential Equation $Y'' - 6Y' + 9Y = 0$ and Its Solutions

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Introduction to the Differential Equation

Differential equations form the backbone of mathematical modeling across various scientific disciplines, including physics, engineering, and biology. Among these, second-order linear homogeneous differential equations are foundational due to their prevalence and the rich theory that supports their solutions. The equation under consideration,

$$
Y'' - 6Y' + 9Y = 0,
$$

is a classic example of such an equation. It is a linear differential equation with constant coefficients, which makes it particularly amenable to standard solution techniques such as characteristic equations.

In this review, we will thoroughly analyze this differential equation, verify the proposed solutions $Y = e^{3x}$ and $Y_2 = xe^{3x}$, and explore their significance within the broader context of solving second-order linear differential equations.

Understanding the Differential Equation

The Nature of the Equation

The equation

$$
Y'' - 6Y' + 9Y = 0
$$

is linear, homogeneous, and has constant coefficients. These characteristics imply that the general solution can be constructed systematically, often through the characteristic equation method.

Key points:

- Linear: The equation involves no nonlinear terms of Y, Y', or Y''.
- Homogeneous: The right side equals zero, indicating solutions can be superimposed.
- Constant coefficients: The coefficients of derivatives are constants, simplifying the solution process.

The Significance of the Equation

This particular differential equation models systems where the response depends linearly on the function and its derivatives, such as mechanical oscillations, electrical circuits, or thermal processes with constant properties.

Methodology for Solving the Differential Equation

The standard approach involves assuming a solution of the form $Y = e^{rx}$, which leads to the characteristic polynomial. This approach exploits the linearity and constant coefficients of the differential equation.

The Characteristic Equation

Suppose $Y = e^{rx}$, then

$$
Y' = r e^{rx}, \quad Y'' = r^2 e^{rx}.
$$

Substituting into the differential equation, we get:

$$
r^2 e^{rx} - 6r e^{rx} + 9 e^{rx} = 0.
$$

Dividing through by e^{rx} (which is never zero):

$$
r^2 - 6r + 9 = 0.
$$

This quadratic equation is the characteristic equation.

Solving the Characteristic Equation

$$
r^2 - 6r + 9 = 0.
$$

Using the quadratic formula:

$$
r = \frac{6 \pm \sqrt{(-6)^2 - 4 \times 1 \times 9}}{2} = \frac{6 \pm \sqrt{36 - 36}}{2} = \frac{6 \pm 0}{2} = 3.
$$

Thus, the discriminant is zero, indicating a repeated root:

$$
r = 3.
$$

This is a critical point because the nature of the roots determines the form of the solution.

General Solution of the Differential Equation

Given the repeated root $r = 3$, the general solution to the differential equation is:

$$
Y(x) = C_1 e^{3x} + C_2 x e^{3x},
$$

where C_1 and C_2 are arbitrary constants.

Explanation:

- The term e^{3x} corresponds to the single root $r = 3$.
- Due to the multiplicity of the root (double root), the second linearly independent solution involves multiplying by x, giving $x e^{3x}$.

This form ensures the solution set forms a basis for the solution space, satisfying linear independence and completeness.

Verification of the Proposed Solutions

The problem statement suggests verifying that:

- $Y_1 = e^{3x}$,
- $Y_2 = x e^{3x}$,

are solutions to the differential equation.

This section will rigorously verify each.

Verification of $Y = e^{3x}$

Step 1: Compute derivatives

$$
Y' = \frac{d}{dx} e^{3x} = 3 e^{3x},
$$
$$
Y'' = \frac{d^2}{dx^2} e^{3x} = 9 e^{3x}.
$$

Step 2: Substitute into the differential equation

$$
Y'' - 6Y' + 9Y = 9 e^{3x} - 6 \times 3 e^{3x} + 9 e^{3x} = 9 e^{3x} - 18 e^{3x} + 9 e^{3x}.
$$

Step 3: Simplify

$$
(9 - 18 + 9) e^{3x} = 0 \times e^{3x} = 0.
$$

Result: The function e^{3x} satisfies the differential equation, confirming it as a solution.

Verification of $Y = x e^{3x}$

Step 1: Compute derivatives

First derivative:

$$
Y' = \frac{d}{dx} (x e^{3x}) = e^{3x} + x \times 3 e^{3x} = e^{3x} + 3x e^{3x} = e^{3x} (1 + 3x).
$$

Second derivative:

$$
Y'' = \frac{d}{dx} \left(e^{3x} (1 + 3x) \right).
$$

Applying the product rule:

$$
Y'' = \frac{d}{dx} e^{3x} \times (1 + 3x) + e^{3x} \times \frac{d}{dx} (1 + 3x).
$$

Calculate derivatives:

$$
\frac{d}{dx} e^{3x} = 3 e^{3x},
$$

$$
\frac{d}{dx} (1 + 3x) = 3.
$$

Therefore,

$$
Y'' = 3 e^{3x} (1 + 3x) + e^{3x} \times 3 = 3 e^{3x} (1 + 3x) + 3 e^{3x} = 3 e^{3x} (1 + 3x + 1) = 3 e^{3x} (2 + 3x).
$$

Step 2: Substitute into the original equation

$$
Y'' - 6Y' + 9Y = 3 e^{3x} (2 + 3x) - 6 \times e^{3x} (1 + 3x) + 9 \times x e^{3x}.
$$

\]

Express all terms explicitly:

\[
= 3 e^{3x} (2 + 3x) - 6 e^{3x} (1 + 3x) + 9 x e^{3x}.
\]

Factor out \(e^{3x} \):

\[
e^{3x} \left[3 (2 + 3x) - 6 (1 + 3x) + 9 x \right].
\]

Simplify inside brackets:

\[
3 \times 2 + 3 \times 3x - 6 \times 1 - 6 \times 3x + 9 x = 6 + 9x - 6 - 18x + 9x.
\]

Combine like terms:

\[
(6 - 6) + (9x - 18x + 9x) = 0 + (0) = 0.
\]

Result: The entire expression simplifies to zero, confirming that \(x e^{3x} \) is a solution.

Implications of the Solution Set

The solutions \(e^{3x} \) and \(x e^{3x} \) form a fundamental set of solutions for the differential equation.

- Linear independence:

The Wronskian determinant \(W(y_1, y_2) \) for these functions is non-zero, ensuring linear independence:

\[
W = y_1 y_2' - y_1' y_2,

\]

where

\[
y_1 = e^{3x}, \quad y_2 = x e^{3x}.
\]

- General solution:

\[
Y(x) = C_1 e^{3x} + C_2 x e^{3x}.
\]

This general solution encompasses all possible solutions to the differential equation, with \(C_1, C_2 \) arbitrary constants determined by initial or boundary conditions

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