Consider The Following Series. [infinity] N = 1 (1)n 1 N32n |error| < 0.0005 Show That The Series

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Introduction

The convergence of infinite series is a fundamental concept in mathematical analysis, with applications spanning calculus, scientific computations, and engineering. Understanding how to determine whether a series converges, and estimating the error in its partial sums, is essential for both theoretical insights and practical calculations. In this article, we analyze the series given by

$$
\sum_{n=1}^{\infty} \frac{(-1)^{n}}{N 3^{2n}}
$$

and demonstrate how to show that the error after summing up to a certain number of terms is less than 0.0005.

Understanding the Series

Series Expression

The series under consideration can be written explicitly as:

$$
\sum_{n=1}^{\infty} \frac{(-1)^n}{N \cdot 3^{2n}}
$$

Note that the notation "N" in the denominator suggests that the terms depend on a variable N. To interpret this correctly, we assume the series is:

$$
\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 3^{2n}}

\]

since the original notation appears to be a typographical or formatting issue, and the common form of such series involves the index n, the term $1/n$, and exponential powers.

Therefore, the series becomes:

$$
\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 3^{2n}}
$$

which is an alternating series with terms decreasing in magnitude, given the exponential growth in the denominator.

Rewriting the Series

Observe that:

$$
3^{2n} = (3^2)^n = 9^n
$$

Thus, the series simplifies to:

$$
\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 9^n}
$$

This form allows us to recognize the series as an alternating series with terms:

$$
a_n = \frac{1}{n \cdot 9^n}
$$

which are positive, decreasing, and tend to zero as $n \to \infty$.

Convergence of the Series

Applying the Alternating Series Test

The series:

$$
\sum_{n=1}^{\infty} (-1)^n a_n
$$

with

$$
a_n = \frac{1}{n \cdot 9^n}
$$

can be analyzed using the Alternating Series Test (Leibniz Criterion). The test states that if:

1. $a_{n+1} \leq a_n$ for all n, i.e., the sequence is decreasing.

2. $\lim_{n \to \infty} a_n = 0$.

then the series converges.

Let's verify these conditions:

- Decreasing sequence: Since $a_n = \frac{1}{n \cdot 9^n}$, as n increases, both n and 9^n increase, making a_n decrease monotonically.

- Limit to zero:

$$
\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n \cdot 9^n} = 0
$$

because exponential growth dominates polynomial growth.

Conclusion: The series converges by the Alternating Series Test.

Estimating the Error in Partial Sums

Partial Sums and Remainder (Error) Term

Suppose we define the partial sum:

$$
S_N = \sum_{n=1}^{N} \frac{(-1)^n}{n \cdot 9^n}
$$

The remainder (or error) after N terms when approximating the sum of the infinite series is given by:

$$
|R_N| = \left| \sum_{n=N+1}^{\infty} \frac{(-1)^n}{n \cdot 9^n} \right|
$$

The Alternating Series Estimation Theorem states that:

$$
|R_N| \leq a_{N+1}
$$

In our case:

$$
|R_N| \leq \frac{1}{(N+1) \cdot 9^{N+1}}
$$

Our goal is to choose N such that:

$$
|R_N| < 0.0005
$$

which ensures the partial sum approximates the total sum within this error bound.

Determining the Number of Terms Needed

Solving for N

We need:

$$
\frac{1}{(N+1) \cdot 9^{N+1}} < 0.0005
$$

Rearranged as:

$$
(N+1) \cdot 9^{N+1} > 2000
$$

This inequality guides us to find the smallest integer N satisfying this condition.

Estimating N

Let's evaluate the expression for increasing N:

- For $N=2$:

$$
(3) \cdot 9^{3} = 3 \times 729 = 2187 > 2000
$$

- For $N=1$:

$$
(2) \cdot 9^{2} = 2 \times 81 = 162 < 2000
$$

- For $N=3$:

$$
4 \times 9^{4} = 4 \times 6561 = 26244 > 2000
$$

But since we need the minimal N, check $N=2$:

$$
(N+1) \cdot 9^{N+1} = 3 \times 729 = 2187 > 2000
$$

which satisfies the inequality.

Thus, choosing $N=2$ suffices:

$$
|R_2| \leq \frac{1}{3 \times 9^{3}} = \frac{1}{2187} \approx 0.000457
$$

which is less than 0.0005.

Therefore, summing the first two terms of the series ensures the error is less than 0.0005.

Explicit Calculation of the Partial Sum S_2

Compute:

$$
S_2 = \frac{(-1)^1}{1 \cdot 9^{1}} + \frac{(-1)^2}{2 \cdot 9^{2}}
$$

\]

which simplifies to:

$$
S_2 = - \frac{1}{9} + \frac{1}{2 \times 81} = - \frac{1}{9} + \frac{1}{162}
$$

Expressing both with common denominator 162:

$$
- \frac{18}{162} + \frac{1}{162} = - \frac{17}{162}
$$

Numerical value:

$$
S_2 \approx -0.104938
$$

Given the error estimate, the true sum of the series differs from S_2 by less than 0.0005.

Conclusion

By analyzing the series

$$
\sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 9^{n}}
$$

we have demonstrated its convergence through the Alternating Series Test. Moreover, we have shown how to estimate the number of terms required to approximate the series within a specified error margin. Specifically, summing the first two terms yields an approximation with an error less than 0.0005, as the remainder after two terms is bounded by

$$
|R_2| \leq \frac{1}{3 \times 9^{3}} \approx 0.000457 < 0.0005.
$$

This process underscores the importance of understanding the decay rate of series terms and the utility of error bounds in practical computations. Such techniques are invaluable in numerical analysis, enabling precise approximations of infinite sums with finite calculations.

Frequently Asked Questions

What is the main goal when analyzing the series ∑ (from n=1 to ∞) of (1/n) in the context of the given problem?

The primary goal is to determine whether the series converges and, if so, to show that its partial sums approximate the series within an error margin of less than 0.0005.

How can the integral test be applied to evaluate the convergence of the series ∑ (1/n)?

The integral test involves evaluating the integral of 1/x from 1 to infinity; since this integral diverges, it indicates that the harmonic series ∑ (1/n) diverges, which is crucial for understanding the series' behavior.

What technique can be used to estimate the partial sums of the series to ensure the error is less than 0.0005?

Using the comparison or the integral test, along with the properties of harmonic series partial sums, helps estimate the sum's accuracy and determine how many terms are needed to achieve the desired error bound.

How does the concept of the alternating series test relate if the series involves alternating signs?

If the series alternates in sign, the alternating series test can be used to confirm convergence and to estimate the error by considering the magnitude of the next term in the series.

What is the significance of choosing N such that |S_N - S| < 0.0005 in the context of this series?

Selecting N ensures that the partial sum S_N approximates the total sum S within the specified error margin, providing a reliable approximation for practical purposes.

Additional Resources

Consider The Following Series. [infinity] N = 1 (1)n 1 N32n |error| < 0.0005 Show That The Series

In the realm of mathematical analysis, series play a crucial role, serving as foundational blocks for understanding convergence, approximation, and the behavior of functions.

Among these, infinite series often pose intriguing challenges: How do we determine whether they converge? If they do, how accurately can we approximate their sum? Today, we delve into one such series, exploring its properties and demonstrating how to approximate its sum within a specified error margin.

Understanding the Series: The Setup and Notation

The series under consideration is expressed as:

$$
\sum_{n=1}^\infty \frac{(1)^n}{n \cdot 3^{2n}}
$$

or more explicitly,

$$
\sum_{n=1}^\infty \frac{1}{n \cdot 3^{2n}}
$$

since $(1)^n = 1$ for all n.

Key features:

- Terms: Each term is $\frac{1}{n \cdot 3^{2n}}$.
- Indexing: The sum extends from $n=1$ to infinity.
- Error Bound: The goal is to approximate the sum with an error less than 0.0005.

Note: The notation in the original statement appears to have some formatting issues, but based on context, the core series involves terms of the form $\frac{1}{n \cdot 3^{2n}}$.

The Significance of Series Convergence

Before attempting to compute or approximate the sum, it's essential to verify whether the series converges. Infinite series that do not converge are not meaningful for summation in the conventional sense.

Convergence Criteria:

- Comparison Test: Comparing the series with a known convergent series.
- Ratio Test: Examining the limit of the ratio of successive terms.
- Root Test: Considering the nth root of the absolute value of terms.

Given the terms involve exponential decay $3^{2n} = (3^2)^n = 9^n$, the series is expected to converge rapidly.

Proving Convergence: The Ratio Test

Let's apply the ratio test to determine convergence:

$$
L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|
$$

where:

$$
a_n = \frac{1}{n \cdot 3^{2n}}
$$

Calculate:

$$
a_{n+1} = \frac{1}{(n+1) \cdot 3^{2(n+1)}} = \frac{1}{(n+1) \cdot 3^{2n + 2}} = \frac{1}{(n+1) \cdot 3^{2n} \cdot 3^2} = \frac{1}{(n+1) \cdot 3^{2n} \cdot 9}
$$

Now, the ratio:

$$
\left| \frac{a_{n+1}}{a_n} \right| = \frac{\frac{1}{(n+1) \cdot 3^{2n} \cdot 9}}{\frac{1}{n \cdot 3^{2n}}} = \frac{n \cdot 3^{2n}}{(n+1) \cdot 3^{2n} \cdot 9} = \frac{n}{(n+1) \cdot 9}
$$

Simplify:

$$
= \frac{n}{9(n+1)} = \frac{1}{9} \cdot \frac{n}{n+1}
$$

As $n \to \infty$:

$$
L = \frac{1}{9} \cdot 1 = \frac{1}{9}
$$

Since $L = \frac{1}{9} < 1$, the series converges absolutely by the ratio test.

Conclusion: The series converges, and since the terms decrease rapidly, we can approximate the sum with finite partial sums.

Strategies for Approximate Summation

Given the convergence, the next goal is to compute the partial sum S_N:

$$
S_N = \sum_{n=1}^N \frac{1}{n \cdot 3^{2n}}
$$

such that the error $|S - S_N| < 0.0005$, where S is the actual sum.

Approach:

1. Identify a suitable N: Determine the number of terms needed so that the tail $R_N = \sum_{n=N+1}^\infty a_n$ is less than 0.0005.
2. Estimate the tail R_N: Use an upper bound to ensure the approximation's accuracy.
3. Compute partial sums: Calculate the sum up to N.

Estimating the Remainder (Tail) of the Series

Because the terms are positive and decreasing, the tail R_N can be bounded by the first omitted term:

$$
R_N < a_{N+1}
$$

This is a standard result for decreasing positive terms.

Recall:

$$
a_{N+1} = \frac{1}{(N+1) 3^{2(N+1)}} = \frac{1}{(N+1) 9^{N+1}}
$$

To ensure $R_N < 0.0005$, we need:

$$
a_{N+1} < 0.0005
$$

which translates to:

$$
\frac{1}{(N+1) 9^{N+1}} < 0.0005
$$

or equivalently,

$$
(N+1) \cdot 9^{N+1} > 2000
$$

Determining the Number of Terms Needed

Let's evaluate a_{N+1} for increasing N:

- For $N=3$:

$$
a_4 = \frac{1}{4 \cdot 9^4} = \frac{1}{4 \cdot 6561} \approx \frac{1}{26244} \approx 0.000038
$$

which is less than 0.0005, so including up to $n=3$ terms yields an error less than 0.0005.

Verification:

- Sum up to $N=3$:

$$
S_3 = \frac{1}{1 \cdot 3^{2}} + \frac{1}{2 \cdot 3^{4}} + \frac{1}{3 \cdot 3^{6}}
$$

Calculate each term:

1. $n=1$:

$$
\frac{1}{1 \cdot 3^{2}} = \frac{1}{1 \cdot 9} = \frac{1}{9} \approx 0.1111
$$

2. $n=2$:

$$
\frac{1}{2 \cdot 3^{4}} = \frac{1}{2 \cdot 81} = \frac{1}{162} \approx 0.00617
$$

3. $n=3$:

$$
\frac{1}{3 \cdot 3^{6}} = \frac{1}{3 \cdot 729} = \frac{1}{2187} \approx 0.000457
$$

Sum:

$$
S_3 \approx 0.1111 + 0.00617 + 0.000457 \approx 0.1179
$$

The next term a_4:

$$

a_4 = \frac{1}{4 \cdot 9^{4}} = \frac{1}{4 \cdot 6561} \approx 0.000038
\]

which is less than 0.0005, confirming that summing up to \(n=3 \) provides an approximation with error less than 0.0005.

Final Approximate Sum and Error Bound

Approximate sum:

\[
\boxed{
S \approx 0.1179
}
\]

Error bound:

\[
|S - S_3| < a_4 \approx 0.000038 < 0.0005
\]

This guarantees that the partial sum up to \(n=3 \) approximates the true sum with an error less than 0.0005.

Alternative Approach: Expressing the Series in Closed-Form

While the summation method above is straightforward, it’s instructive to explore whether the series can be expressed in a closed form. Recognizing the pattern of the terms suggests a relation to the Taylor expansion of certain functions.

Recall the Taylor series expansion of the natural logarithm:

\[
- \ln(1 - x) = \sum_{n=1}^\infty \frac{x^n}{n}
\]

for \(|x| < 1 \).

Our series involves \(1/(n \cdot 9^{n})

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Lt 0 0005 Show That The Series

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