Consider The Initial Value Problem Dy/dt = }t|+y, Y(0)=0.what Does The Fundamental Existence Theorem

Consider The Initial Value Problem Dy/dt = }t|+y, Y(0)=0.what Does The Fundamental Existence Theorem

When working with differential equations, understanding whether solutions exist and whether they are unique is fundamental to analyzing the behavior of systems modeled by these equations. The initial value problem (IVP) specified as Dy/dt = }t|+y, with Y(0) = 0, presents an interesting case for discussion, especially regarding the applicability of the Fundamental Existence Theorem. This theorem provides critical insights into whether solutions to differential equations exist near a given initial point and whether these solutions are unique. In this article, we will explore the initial value problem in question, interpret the components of the differential equation, and delve into the implications of the Fundamental Existence Theorem.

Understanding the Initial Value Problem Dy/dt = }t|+y, Y(0)=0

Analyzing the Differential Equation

The differential equation under consideration is:

\[\frac{dy}{dt} = }t| + y \]

along with the initial condition:

\[Y(0) = 0 \]

At first glance, the notation }t| might seem unusual or potentially a typo. Typically, in differential equations, the right-hand side is expressed as a function of \(t \) and \(y \). Assuming that the notation }t| is intended to represent \(|t| \), the absolute value of \(t \), the differential equation can be written as:

\[\frac{dy}{dt} = |t| + y \]

with the initial condition:

\[y(0) = 0 \]

This form is more standard and allows us to analyze the problem using well-established theorems.

Components of the Equation

- The right-hand side $f(t, y) = |t| + y$: This function depends on both t and y, and is composed of the absolute value function $|t|$ and the linear term y.
- Initial condition $y(0) = 0$: Specifies the value of the solution at $t=0$.

Understanding the properties of $f(t, y)$ is essential for applying the Fundamental Existence Theorem.

The Fundamental Existence and Uniqueness Theorem

Statement of the Theorem

The Fundamental Existence and Uniqueness Theorem (also known as Picard-Lindelöf Theorem) states that:

> If $f(t, y)$ is continuous on a rectangle $R = \{ (t, y) : |t - t_0| \leq a, |y - y_0| \leq b \}$, and if f satisfies a Lipschitz condition with respect to y on R, then there exists an interval I around t_0 on which the initial value problem:
>
> $$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0$$
>
> has a unique solution.

Key points:
- Continuity of $f(t, y)$: Ensures solutions exist.
- Lipschitz condition in y: Ensures solutions are unique.

Implications for Our Problem

To determine whether the theorem applies, we need to check:
- If $f(t, y) = |t| + y$ is continuous near $t=0$ and $y=0$.
- If $f(t, y)$ satisfies the Lipschitz condition with respect to y.

Analyzing the Conditions for the Given Differential Equation

Continuity of $f(t, y) = |t| + y$

- The function $|t|$ is continuous everywhere.
- The function y is continuous everywhere.
- The sum of continuous functions is continuous.

Conclusion: $f(t, y)$ is continuous for all (t, y), including at $t=0$, $y=0$.

Lipschitz Condition in y

- $f(t, y) = |t| + y$ is linear in y, with a coefficient of 1.
- The Lipschitz constant with respect to y can be taken as 1, since:

$$|f(t, y_1) - f(t, y_2)| = |(|t| + y_1) - (|t| + y_2)| = | y_1 - y_2 |$$

- Therefore, f is Lipschitz continuous in y uniformly in t.

Conclusion: The Lipschitz condition is satisfied with Lipschitz constant $L = 1$.

Applying the Fundamental Theorem to Our Problem

Existence of a Solution Near $t=0$

Given the continuity and Lipschitz conditions are satisfied near $(t, y) = (0, 0)$, the theorem guarantees the existence of at least one solution $y(t)$ passing through $y(0) = 0$.

- Interval of existence: There exists some interval $|t| \leq h$, where $h > 0$, such that the solution exists and is unique.

Uniqueness of the Solution

The Lipschitz condition with respect to y ensures that the solution is unique within the interval of existence.

Explicit Solution and Further Analysis

Solving the Differential Equation

The differential equation:

$$\frac{dy}{dt} = |t| + y$$

can be approached by considering cases:

- For $t \geq 0$:

$$\frac{dy}{dt} = t + y$$

- For $t < 0$:

$$\frac{dy}{dt} = -t + y$$

Using integrating factors, we can find explicit solutions.

Solution for $t \geq 0$

The integrating factor:

$$\mu(t) = e^{-\int 1 \, dt} = e^{-t}$$

Multiplying both sides:

$$e^{-t} \frac{dy}{dt} - e^{-t} y = t e^{-t}$$

which simplifies to:

$$\frac{d}{dt} \left(y e^{-t} \right) = t e^{-t}$$

Integrate both sides:

$$y e^{-t} = \int t e^{-t} dt + C$$

The integral:

$$\int t e^{-t} dt$$

can be computed via integration by parts:

Let $u = t$, $dv = e^{-t} dt$

then:

$$du = dt$$
$$v = - e^{-t}$$

Thus:

$$\int t e^{-t} dt = - t e^{-t} + \int e^{-t} dt = - t e^{-t} - e^{-t} + D$$

Putting everything together:

$$y e^{-t} = - t e^{-t} - e^{-t} + C$$

Multiply both sides by e^{t}:

$$y = - t - 1 + C e^{t}$$

Applying initial condition $y(0) = 0$:

$$0 = - 0 - 1 + C e^{0} \Rightarrow 0 = -1 + C \Rightarrow C = 1$$

Solution for $t \geq 0$:

$$y(t) = - t - 1 + e^{t}$$

Similarly, for $t < 0$:

$$\frac{dy}{dt} = - t + y$$

The integrating factor:

$$\mu(t) = e^{- \int 1 dt} = e^{-t}$$

Proceeding similarly:

$$\frac{d}{dt} \left(y e^{-t} \right) = - t e^{-t}$$

Integral:

$$\int - t e^{-t} dt$$

Using integration by parts:

$$u = - t, \quad dv = e^{-t} dt$$
$$du = - dt, \quad v = - e^{-t}$$

$$\int - t e^{-t} dt = (- t)(- e^{-t}) - \int - e^{-t} (- dt) = t e^{-t} - \int e^{-t} dt = t e^{-t} - (- e^{-t}) = t e^{-t} + e^{-t} + D$$

Thus:

$$y e^{-t} = t e^{-t} + e^{-t} + C$$

Multiply both sides by e^{t}:

$$y = t + 1 + C e^{t}$$

Applying $y(0) = 0$:

$$0 = 0 + 1 + C \Rightarrow C = -1$$

Solution for

Frequently Asked Questions

What is the initial value problem given in the context?

The initial value problem is Dy/dt = |t| + y, with Y(0) = 0.

What does the Fundamental Existence Theorem state regarding differential equations?

It states that if the function f(t, y) is continuous and satisfies a Lipschitz condition in y near a point, then there exists a unique local solution to the initial value problem around that point.

Is the function f(t, y) = |t| + y continuous at all points?

Yes, since |t| is continuous everywhere and y is a variable, f(t, y) = |t| + y is continuous for all (t, y).

Does the function f(t, y) satisfy the Lipschitz condition in y?

Yes, since f(t, y) is linear in y with a constant coefficient, it satisfies the Lipschitz condition in y with Lipschitz constant 1.

What does the Fundamental Existence Theorem guarantee for this initial value problem?

It guarantees a unique local solution Y(t) exists near t=0 because f(t, y) is continuous and Lipschitz in y around the initial point.

Is the initial value problem well-posed according to the theorem?

Yes, because the conditions for the theorem are satisfied, ensuring existence and uniqueness of the solution near t=0.

What is the significance of the initial condition Y(0)=0 in applying the theorem?

It provides the specific point around which the theorem guarantees a local solution, ensuring the conditions are checked at t=0, y=0.

Can the solution be extended beyond the initial interval guaranteed by the theorem?

Yes, if the solution remains bounded and the conditions on f(t, y) hold, it can often be extended further, but the theorem guarantees local existence.

How does the absolute value function |t| affect the application

of the theorem?

Since |t| is continuous everywhere, it does not violate the continuity requirement, allowing the theorem to be applied without issue.

Additional Resources

Consider The Initial Value Problem Dy/dt = |t| + y, Y(0) = 0: An In-Depth Analysis of Existence and Uniqueness

When examining the initial value problem (IVP) given by

$$ \frac{dy}{dt} = |t| + y, \quad y(0) = 0, $$

a fundamental question arises: does a solution exist, and is it unique? This inquiry touches upon core concepts in differential equations, particularly the Fundamental Existence Theorem (also known as the Picard–Lindelöf theorem). Understanding this theorem’s application to the IVP provides insight into the behavior of solutions, their existence, and conditions for uniqueness. This article aims to dissect the problem thoroughly, analyze the theorem's implications, and explore the nuances of solutions to this specific differential equation.

Understanding the Initial Value Problem

Before delving into the theorem itself, let's understand the structure of the problem:

$$ \frac{dy}{dt} = |t| + y, \quad y(0) = 0. $$

This is a first-order linear differential equation with a non-smooth (absolute value) term. The initial condition specifies that at $t=0$, the solution y must be zero.

Key features of this problem include:
- The presence of the absolute value function $|t|$, which introduces a point of non-differentiability at $t=0$.
- The linearity in y, which typically simplifies analysis.
- The initial condition at $t=0$, exactly where the function $|t|$ is non-differentiable.

Understanding how the absolute value impacts the existence and uniqueness of solutions is central to the discussion.

The Fundamental Existence and Uniqueness Theorem

Statement of the Theorem

The Fundamental Existence Theorem (or Picard–Lindelöf theorem) states:

Given an initial value problem

$$\frac{dy}{dt} = f(t, y), \quad y(t_0) = y_0,$$

if $f(t, y)$ and $\partial f / \partial y$ are continuous in some rectangle $R = \{ (t, y) : |t - t_0| \leq a, |y - y_0| \leq b \}$, then there exists an interval $I \subseteq \mathbb{R}$, containing t_0, on which a unique solution $y(t)$ exists.

Features of the theorem:
- Continuity of $f(t, y)$ is crucial for existence.
- Continuity of the partial derivative $\partial f / \partial y$ ensures uniqueness.
- The theorem guarantees local solutions; global solutions require additional conditions.

Application to Our Problem

In our problem:

$$f(t, y) = |t| + y.$$

- At $t=0$, $f(t, y)$ is continuous in y and t, since $|t|$ is continuous everywhere.
- The partial derivative with respect to y is:

$$\frac{\partial f}{\partial y} = 1,$$

which is constant and continuous everywhere.

Given these observations, the fundamental theorem suggests that there exists a unique solution passing through $(0, 0)$ in some interval around $t=0$.

Analyzing the Solution Near $t=0$

Existence of a Solution

Since $f(t, y) = |t| + y$ is continuous in a neighborhood of $(0, 0)$, the theorem assures us of at least a local solution. This solution exists in some interval $(-\delta, \delta)$ for a small $\delta > 0$.

Moreover, because $\partial f / \partial y = 1$ is continuous everywhere, the solution is unique in this interval.

Explicit Solutions and Their Behavior

Let's attempt to find an explicit solution to better understand the behavior:

$$\frac{dy}{dt} - y = |t|.$$

This is a linear ODE with integrating factor e^{-t}:

$$y(t) e^{-t} = \int |t| e^{-t} dt + C.$$

Given the absolute value, we need to split the integral:

- For $t \geq 0$:

$$|t| = t,$$

and

$$y(t) e^{-t} = \int t e^{-t} dt + C.$$

- For $t < 0$:

$$|t| = -t,$$

and

$$y(t) e^{-t} = \int -t e^{-t} dt + C.$$

Calculating these integrals:

For $t \geq 0$:

$$
\int t e^{-t} dt.
$$

Using integration by parts:

Let $u = t \Rightarrow du = dt$,
$dv = e^{-t} dt \Rightarrow v = - e^{-t}$.

Then:

$$
\int t e^{-t} dt = - t e^{-t} + \int e^{-t} dt = - t e^{-t} - e^{-t} + D,
$$

where D is an integration constant.

Thus,

$$
y(t) e^{-t} = - t e^{-t} - e^{-t} + C,
$$

leading to:

$$
y(t) = - t - 1 + C e^{t}.
$$

Applying initial condition $y(0) = 0$:

$$
0 = - 0 - 1 + C e^{0} \Rightarrow 0 = -1 + C \Rightarrow C = 1.
$$

Therefore, for $t \ge 0$:

$$
\boxed{
y(t) = - t - 1 + e^{t}.
}
$$

For $t < 0$:

$$
\int - t e^{-t} dt.
$$

Similarly, set $u = - t \Rightarrow du = - dt$:

$$
\int u e^{u} du,
$$

because:

$$
- t e^{-t} dt = u e^{u} du,
$$

and

$$
\int u e^{u} du.
$$

Using integration by parts:

Let $u = u \Rightarrow du = du$,
$dv = e^{u} du \Rightarrow v = e^{u}$.

Then:

$$
\int u e^{u} du = u e^{u} - \int e^{u} du = u e^{u} - e^{u} + D.
$$

Substituting back $u = - t$:

$$
\int - t e^{-t} dt = (- t) e^{-t} - e^{-t} + D.
$$

Thus,

$$
y(t) e^{-t} = (- t) e^{-t} - e^{-t} + C,
$$

which simplifies to:

$$
y(t) = - t - 1 + C e^{t}.
$$

Applying initial condition at $t=0$:

$$
0 = - 0 - 1 + C e^{0} \Rightarrow C = 1,
$$

matching the solution for $t \geq 0$.

Summary of the explicit solution:

$$
\boxed{
y(t) = - t - 1 + e^{t}, \quad t \text{ in a neighborhood of } 0.
}
$$

This solution is smooth for all t, and satisfies the initial condition.

Implications of the Solution and the Theorem

The explicit solution confirms the theorem's prediction of existence and uniqueness near \(t=0 \). The smoothness of \(y(t) \) indicates that the initial value problem is well-posed locally.

Key features:
- The solution is valid in a small interval around zero.
- The initial condition is satisfied.
- The solution is unique in this interval.

Pros of Applying the Fundamental Existence Theorem:

- Guarantees a solution exists without explicitly solving the equation.
- Ensures the solution is unique, providing stability to the problem.
- Applicable even when the differential equation involves non-smooth functions like \(|t| \), as long as continuity is maintained.

Cons or Limitations:

- The theorem guarantees local existence, not necessarily global.
- The solution's behavior outside the interval of existence may be complex or undefined.
- Non-differentiability at certain points (like \(t=0 \) for \(|t| \)) can complicate extending solutions globally.

Addressing Non-Differentiability at \(t=0 \)

Since \(|t| \) is not differentiable at \(t=0 \), a natural concern is whether this affects the existence and uniqueness. The key points are:

- The fundamental theorem requires continuity of \(f(t, y) \), which holds here.
- The partial derivative \(\partial f

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Consider The Initial Value Problem Dy Dt T Y Y 0 0 What Does The Fundamental Existence Theorem

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