Consider The Vector Field F(x, Y, Z) = (-y, -x, 8z). Show That F Is A Gradient Vector Field F = V By

Consider The Vector Field F(x, Y, Z) = (-y, -x, 8z). Show That F Is A Gradient Vector Field F = V By

Understanding whether a vector field is a gradient field is fundamental in vector calculus, as it relates to the potential functions and the conservative nature of forces. In this article, we will explore the vector field $F(x, y, z) = (-y, -x, 8z)$ and demonstrate that it indeed represents a gradient vector field, i.e., $F = \nabla V$ for some scalar potential function $V(x, y, z)$. We will provide a comprehensive, step-by-step analysis to establish this property, along with the relevant theoretical background, calculations, and implications.

Introduction to Gradient Vector Fields

Before diving into the specific vector field, it is crucial to understand what constitutes a gradient vector field.

What is a Gradient Vector Field?

A vector field F in three-dimensional space is called a gradient field (or conservative vector field) if there exists a scalar function $V(x, y, z)$ such that:

$$ \mathbf{F} = \nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right) $$

This means F is the gradient of some potential function V.

Characteristics of Gradient Fields

- Curl-free: Gradient fields have zero curl, i.e., $\nabla \times \mathbf{F} = \mathbf{0}$

- Path independence: Line integrals of F are path-independent within simply connected regions.

- Existence of potential function: There exists a scalar potential V such that $F = \nabla V$.

Analyzing the Given Vector Field F(x, y, z) = (-y, -x, 8z)

Let's analyze the vector field:

$$ \mathbf{F}(x, y, z) = (-y, -x, 8z) $$

Our goal is to determine whether F can be expressed as the gradient of some scalar potential function V(x, y, z).

Step 1: Expressing the Gradient Conditions

Suppose F is a gradient field, then there exists V(x, y, z) such that:

$$
\begin{cases}
\frac{\partial V}{\partial x} = -y \\
\frac{\partial V}{\partial y} = -x \\
\frac{\partial V}{\partial z} = 8z
\end{cases}
$$

Our task reduces to finding V(x, y, z) satisfying these partial derivatives.

Step 2: Integrating Partial Derivatives

Let's integrate each component to find V:

1. From $\frac{\partial V}{\partial x} = -y$:

$$
V(x, y, z) = -y x + h(y, z)
$$

where $h(y, z)$ is an arbitrary function of y and z.

2. From $\frac{\partial V}{\partial y} = -x$:

Differentiate the expression for V with respect to y:

$$
\frac{\partial V}{\partial y} = -x + \frac{\partial h(y, z)}{\partial y}
$$

Set equal to the given partial derivative:

$$
- x + \frac{\partial h(y, z)}{\partial y} = -x
$$

which implies:

$$
\frac{\partial h(y, z)}{\partial y} = 0
$$

Therefore, $h(y, z)$ does not depend on y, so:

$$
h(y, z) = h(z)
$$

3. From $\frac{\partial V}{\partial z} = 8z$:

Differentiate the current form of V with respect to z:

$$
\frac{\partial V}{\partial z} = \frac{\partial}{\partial z} (- y x + h(z)) = \frac{\partial h(z)}{\partial z}
$$

Set equal to the given derivative:

$$
\frac{\partial h(z)}{\partial z} = 8z
$$

Integrate:

$$
h(z) = 4z^2 + C
$$

where C is an arbitrary constant.

Step 3: Constructing the Potential Function V

Putting everything together:

$$
V(x, y, z) = - y x + 4z^2 + C
$$

Since the constant C does not affect the gradient, we can omit it for simplicity.

Verification: Computing the Gradient of V

Let's verify that F equals ∇V:

$$
\nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right) = (- y, - x, 8z)
$$

which matches the original vector field F.

Therefore, the vector field $\mathbf{F}(x, y, z) = (- y, - x, 8z)$ is indeed a gradient field with potential function:

$$
V(x, y, z) = - y x + 4z^2
$$

Implications and Significance

Establishing that F is a gradient vector field has several important implications:

- Conservative Force Field: The field is conservative, meaning the work done in moving along a path depends only on the endpoints, not the path itself.

- Potential Energy Function: The scalar potential V can be interpreted as potential energy in physics or potential in mathematical contexts.

- Zero Curl: Since F is a gradient field, its curl is zero:

$$
\nabla \times \mathbf{F} = \mathbf{0}
$$

which can be verified explicitly for completeness.

Verifying the Curl of F

Compute:

$$
\nabla \times \mathbf{F} = \left(\frac{\partial (8z)}{\partial y} - \frac{\partial (-x)}{\partial z}, \; \frac{\partial (-y)}{\partial z} - \frac{\partial (8z)}{\partial x}, \; \frac{\partial (-x)}{\partial y} - \frac{\partial (-y)}{\partial x} \right)
$$

Calculations:

- $\frac{\partial (8z)}{\partial y} = 0$

- $\frac{\partial (-x)}{\partial z} = 0$

- $\frac{\partial (-y)}{\partial z} = 0$

- $\frac{\partial (8z)}{\partial x} = 0$

- $\frac{\partial (-x)}{\partial y} = 0$

- $\frac{\partial (-y)}{\partial x} = 0$

So,

$$
\nabla \times \mathbf{F} = (0 - 0, 0 - 0, 0 - 0) = \mathbf{0}
$$

which confirms the curl-free nature of F.

Conclusion

In conclusion, the vector field $\mathbf{F}(x, y, z) = (- y, - x, 8z)$ is a gradient vector field, as it can be expressed as the gradient of the scalar potential function:

$$
V(x, y, z) = - y x + 4z^2
$$

This demonstrates that F is conservative, curl-free, and possesses a potential function, fulfilling the criteria for a gradient vector field. Recognizing such properties is essential in various scientific and engineering applications, including physics, where potential functions describe energy landscapes, and in mathematics, where they facilitate the evaluation of line integrals and the analysis of vector fields.

Additional Remarks

- Region of Definition: The above derivation assumes the domain is simply connected, which is typical in Euclidean space $\mathbb{R}^3$.

- Applications: Gradient fields are encountered in gravitational and electrostatic fields, fluid flow, and potential theory.

- Further Study: To deepen understanding, explore conditions under which vector fields are conservative, the role of curl and divergence, and the applications of gradient fields in various physical contexts.

In summary, by integrating the partial derivatives of the vector field, verifying the curl, and constructing the potential function, we have shown that F(x, y, z) = (- y, - x, 8z) is indeed a gradient vector field with potential $V(x, y, z) = - y x + 4z^2$.

Frequently Asked Questions

What is the given vector field F(x, y, z) in the problem?

The vector field is F(x, y, z) = (-y, -x, 8z).

What does it mean for a vector field to be a gradient vector field?

A vector field is a gradient vector field if it can be expressed as the gradient of some scalar potential function $V(x, y, z)$.

How do we verify if F is a gradient vector field?

We verify if there exists a scalar function V such that $F = \nabla V$ by checking if the components of F match the partial derivatives of V.

What are the necessary conditions for F to be a gradient vector field?

The necessary conditions include that the curl of F must be zero (i.e., F is conservative) and that F is defined on a simply connected domain.

How do you find the potential function V for the vector field F?

Integrate each component of F with respect to its corresponding variable, ensuring consistency across mixed partial derivatives to determine V.

What is the potential function V(x, y, z) for F(x, y, z) = (-y, -x, 8z)?

A suitable potential function is $V(x, y, z) = -xy + 4z^2 + C$, where C is an arbitrary constant.

How do you verify that F = ∇V with the found potential function?

Calculate the gradient of V and confirm that it matches the components of F: $\nabla V = (-y, -x, 8z)$.

Why is it important to check the curl of F when establishing it as a gradient field?

Because a vector field is a gradient (or conservative) field if and only if its curl is zero, ensuring path independence of line integrals.

Can the vector field F(x, y, z) = (-y, -x, 8z) be non-conservative? Why or why not?

No, because its curl is zero in a simply connected domain, indicating it is conservative and thus a gradient vector field.

Additional Resources

Consider The Vector Field F(x, y, z) = (-y, -x, 8z). Show That F Is A Gradient Vector Field F = ∇V

Understanding whether a vector field is a gradient vector field is a fundamental aspect of vector calculus, with wide applications in physics, engineering, and mathematics. In this article, we will thoroughly analyze the vector field F(x, y, z) = (-y, -x, 8z) and demonstrate how to determine if it can be expressed as the gradient of some scalar potential function V(x, y, z). This process involves examining the properties of F, applying the conditions for gradient fields, and explicitly constructing the potential function if possible.

Introduction to Gradient Vector Fields

Before diving into the specific vector field F, it's essential to understand what a gradient vector field is. In vector calculus, a vector field F is called a gradient vector field (or conservative field) if there exists a scalar potential function V(x, y, z) such that:

$$
\mathbf{F} = \nabla V = \left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right)
$$

Key Characteristics of Gradient Fields

- Curl-Free: Gradient fields are irrotational; that is, their curl is zero everywhere in the domain.
- Path Independence: The line integral of F between two points depends only on the endpoints, not on the path taken.
- Existence of a Potential: There exists a scalar function V whose gradient yields F.

Step 1: Examine the Vector Field F(x, y, z) = (-y, -x, 8z)

The given vector field is:

$$
\mathbf{F}(x, y, z) = (-y, -x, 8z)
$$

This field assigns a vector to each point in $\mathbb{R}^3$ with components depending on the coordinates. To verify if F is a gradient field, we'll check the necessary conditions.

Step 2: Verify the Curl Condition

A key property of gradient fields is that their curl must be zero everywhere in the domain.

The curl of F is computed as:

\[
\nabla \times \mathbf{F} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_x & F_y & F_z
\end{vmatrix}
\]

Plugging in the components:

\[
F_x = -y,\quad F_y = -x,\quad F_z = 8z
\]

Calculating each component:

Curl in the x-direction, \((\nabla \times \mathbf{F})_x\):

\[
\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = \frac{\partial (8z)}{\partial y} - \frac{\partial (-x)}{\partial z} = 0 - 0 = 0
\]

Curl in the y-direction, \((\nabla \times \mathbf{F})_y\):

\[
\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = \frac{\partial (-y)}{\partial z} - \frac{\partial (8z)}{\partial x} = 0 - 0 = 0
\]

Curl in the z-direction, \((\nabla \times \mathbf{F})_z\):

\[
\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = \frac{\partial (-x)}{\partial x} - \frac{\partial

(-y)}{\partial y} = -1 - (-1) = 0
\]

Result: The curl of F is zero everywhere in \(\mathbb{R}^3\).

\[
\nabla \times \mathbf{F} = \mathbf{0}
\]

Conclusion: Since the curl is zero, F satisfies the irrotational criterion for being a gradient field, assuming the domain is simply connected (which \(\mathbb{R}^3\) is).

Step 3: Constructing the Potential Function V

Having established that F is curl-free, the next step is to explicitly find a scalar potential function V(x, y, z) such that:

\[
\nabla V = \mathbf{F}
\]

which implies:

\[
\frac{\partial V}{\partial x} = -y,\quad
\frac{\partial V}{\partial y} = -x,\quad
\frac{\partial V}{\partial z} = 8z
\]

Step 3.1: Integrate \(F_x = -y\) with respect to \(x\):

\[
V(x, y, z) = \int -y\, dx = -y \cdot x + \phi(y, z)
\]

where \(\phi(y, z)\) is an arbitrary function of \(y\) and \(z\).

Step 3.2: Differentiate \(V\) with respect to \(y\) and compare to \(F_y\):

\[
\frac{\partial V}{\partial y} = -x + \frac{\partial \phi(y, z)}{\partial y}

\]

Set equal to $F_y = -x$:

$$
-x + \frac{\partial \phi(y, z)}{\partial y} = -x \implies \frac{\partial \phi(y, z)}{\partial y} = 0
$$

So, $\phi(y, z)$ is independent of y, i.e., $\phi(y, z) = \psi(z)$.

Step 3.3: Differentiate V with respect to z and compare to F_z:

$$
\frac{\partial V}{\partial z} = \frac{\partial \psi(z)}{\partial z} = 8z
$$

Integrate with respect to z:

$$
\psi(z) = 4z^2 + C
$$

where C is an arbitrary constant.

Final Potential Function

Putting it all together, the potential function $V(x, y, z)$ is:

$$
V(x, y, z) = -x y + 4 z^2 + C
$$

Since adding a constant does not affect the gradient, we can set $C=0$ for simplicity.

Summary and Implications

- The vector field $F(x, y, z) = (-y, -x, 8z)$ satisfies the curl-free condition, which is necessary for being a gradient vector field in simply connected domains.
- The explicit potential function $V(x, y, z) = -x y + 4 z^2$ has been constructed such that \(\nabla V =

\mathbf{F}\).

Therefore, F is a gradient vector field, $F = \nabla V$, with the potential function $V(x, y, z) = -x y + 4 z^2$.

Additional Insights: Why Is This Important?

Recognizing gradient fields allows us to understand the underlying potential energy in physical systems, analyze conservative forces, and compute line integrals with ease. When encountering a vector field like F, verifying the curl condition and explicitly constructing V can reveal its conservative nature and facilitate further analysis.

Final Remarks

This detailed walkthrough demonstrates the systematic approach to verifying whether a vector field is a gradient field:

1. Check the curl of F.
2. Confirm the domain is simply connected.
3. Integrate component-wise to find V.
4. Verify the partial derivatives match the components of F.

By mastering these steps, you're equipped to analyze similar vector fields and deepen your understanding of the fundamental concepts in vector calculus.

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