Demonstrate The Easiest Way To Calculate The Divergence Of The Vector Function C(R)= (r 2+a 2) 3/2r

Demonstrate The Easiest Way To Calculate The Divergence Of The Vector Function C(R)= (r 2+a 2) 3/2r

Calculating the divergence of a vector function is a fundamental operation in vector calculus, with applications spanning physics, engineering, and mathematical analysis. When the vector function is expressed in a specific form involving radial symmetry, such as C(R)= $(r^2 + a^2)^{3/2}$ r, understanding the most straightforward method to compute its divergence becomes essential. In this article, we will explore the step-by-step process to evaluate the divergence of this particular vector function efficiently, emphasizing clarity and practical techniques suitable for students and professionals alike.

Understanding the Vector Function C(R)

Before diving into the calculation, it’s crucial to understand the structure and components of the given vector function.

Definition of the Function

The vector function is given by:

$$C(\mathbf{R}) = (r^2 + a^2)^{3/2} \mathbf{r}$$

where:
- $\mathbf{R} = (x, y, z)$ is the position vector in three-dimensional space.
- $r = |\mathbf{R}| = \sqrt{x^2 + y^2 + z^2}$ is the magnitude of the position vector.
- a is a constant (real number).

This form indicates a radially symmetric vector field, since it depends on the magnitude r and points in the radial direction $\mathbf{r}$.

Significance of Radial Symmetry

Radial symmetry simplifies many vector calculus operations because the field’s components depend solely on the distance from the origin, reducing the problem to derivatives with respect to r.

Fundamental Concepts for Calculating Divergence

The divergence of a vector field $\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$ is defined as:

$$\nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

When the vector field is expressed in terms of spherical symmetry, it is often more efficient to use the divergence formula in spherical coordinates.

Spherical Coordinates and Divergence

In spherical coordinates (r, θ, ϕ):

$$
\begin{cases}
x = r \sin \theta \cos \phi \\
y = r \sin \theta \sin \phi \\
z = r \cos \theta
\end{cases}
$$

and the divergence of a radially symmetric vector field $\mathbf{A} = A_r(r) \hat{\mathbf{r}}$ is:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r(r))$$

This formula greatly simplifies the calculation for fields pointing radially outward or inward.

Step-by-Step Calculation of the Divergence

Given the structure of $C(\mathbf{R})$, it’s best to express it in a form compatible with the divergence formula in spherical coordinates.

Step 1: Express $C(\mathbf{R})$ in terms of r and $\hat{\mathbf{r}}$

Since $\mathbf{r} = r \hat{\mathbf{r}}$, we can write:

$$C(\mathbf{R}) = (r^2 + a^2)^{3/2} r \hat{\mathbf{r}}$$

or equivalently,

$$C(\mathbf{R}) = A(r) \hat{\mathbf{r}}$$

where

$$ A(r) = r (r^2 + a^2)^{3/2} $$

This form makes it clear that the vector field is purely radial with magnitude $A(r)$.

Step 2: Recognize the form suitable for divergence in spherical coordinates

The divergence for a purely radial field:

$$ C(\mathbf{R}) = A(r) \hat{\mathbf{r}} $$

is given by:

$$ \nabla \cdot C = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 A(r) \right) $$

This formula reduces the divergence calculation to a single-variable derivative, simplifying the process.

Step 3: Compute $r^2 A(r)$

Calculate:

$$ r^2 A(r) = r^2 \times r (r^2 + a^2)^{3/2} = r^3 (r^2 + a^2)^{3/2} $$

This expression is the core component of the divergence calculation.

Step 4: Differentiate $r^2 A(r)$ with respect to r

Applying the product rule:

$$
\frac{\partial}{\partial r} \left(r^3 (r^2 + a^2)^{3/2} \right) = \frac{\partial}{\partial r} \left(r^3 \right) \times (r^2 + a^2)^{3/2} + r^3 \times \frac{\partial}{\partial r} \left((r^2 + a^2)^{3/2} \right)
$$

Calculate each term separately:

- $\frac{\partial}{\partial r} r^3 = 3 r^2$
- $\frac{\partial}{\partial r} (r^2 + a^2)^{3/2} = \frac{3}{2} (r^2 + a^2)^{1/2} \times 2 r = 3 r (r^2 + a^2)^{1/2}$

Putting these together:

$$
\frac{\partial}{\partial r} \left(r^3 (r^2 + a^2)^{3/2} \right) = 3 r^2 (r^2 + a^2)^{3/2} + r^3 \times 3 r (r^2 + a^2)^{1/2}
$$

which simplifies to:

$$
3 r^2 (r^2 + a^2)^{3/2} + 3 r^4 (r^2 + a^2)^{1/2}
$$

Step 5: Divide by r^2 to find divergence

Recall:

$$ \nabla \cdot C = \frac{1}{r^2} \times \left[3 r^2 (r^2 + a^2)^{3/2} + 3 r^4 (r^2 + a^2)^{1/2} \right] $$

Dividing through:

$$
\nabla \cdot C = 3 (r^2 + a^2)^{3/2} + 3 r^2 (r^2 + a^2)^{1/2}
$$

Factoring out common terms:

$$
\nabla \cdot C = 3 (r^2 + a^2)^{1/2} \left[(r^2 + a^2) + r^2 \right]
$$

which simplifies to:

$$
\nabla \cdot C = 3 (r^2 + a^2)^{1/2} (2 r^2 + a^2)
$$

Final Result and Interpretation

The divergence of the given vector function $C(\mathbf{R}) = (r^2 + a^2)^{3/2} \mathbf{r}$ is:

$$ \boxed{
\nabla \cdot C = 3 \sqrt{r^2 + a^2} \, (2 r^2 + a^2)
} $$

This expression provides a compact, explicit formula for the divergence at any point $\mathbf{R}$.

Special Cases and Insights

- When $a = 0$, the divergence reduces to:

$$ \nabla \cdot C = 3 r (2 r^2) = 6 r^3 $$

- As $r \to \infty$, divergence grows roughly as $r^{2.5}$, reflecting the increasing magnitude of the vector field.

Practical Applications and Significance

Understanding how to compute divergence for such radial fields is crucial in various physical contexts, such as:

- Electromagnetic fields, where divergence relates to charge density via Gauss's law.
- Fluid dynamics, describing flow divergence or convergence.
- Gravitational fields, where divergence relates to mass density distributions.

Having a straightforward formula like the one derived simplifies analysis and helps in solving real-world problems efficiently.

Summary of Key Steps

To summarize, the easiest approach involves: