Determine The Domain And Range Of (g F)(x) If F Of X Is Equal To 9 Over The Quantity X Squared Minus

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Understanding the domain and range of functions is fundamental in mathematics, especially when analyzing composite functions such as (g ∘ F)(x). The process involves examining how the inner function F(x) behaves and how it influences the overall behavior of the composite function. In this article, we will explore the steps to determine the domain and range of (g ∘ F)(x) when F(x) is given as 9 over the quantity x squared minus a constant, which is a common form in rational functions.

This topic is crucial for students and professionals working in fields related to mathematics, engineering, physics, and economics, where understanding the possible input and output values of functions helps in modeling real-world scenarios accurately. By delving into the specific case where F(x) = 9 / (x^2 - c), we will learn how to handle the restrictions imposed by denominators and how to analyze the resulting composite function comprehensively.

Understanding the Components of the Problem

What Is a Composite Function?

A composite function, denoted as (g ∘ F)(x), is formed by applying one function, g, to the output of another function, F. Mathematically, it is expressed as:

$$ (g \circ F)(x) = g(F(x)) $$

To analyze the domain and range of the composite, we need to understand both the inner function F(x) and the outer function g(x). The overall domain depends on the set of x-values for which F(x) is defined and for which g(F(x)) is also defined. Similarly, the range depends on the possible outputs of the composite function.

Examining the Inner Function F(x) = 9 / (x^2 - c)

Properties of the Rational Function

The given inner function is a rational function:

$$ F(x) = \frac{9}{x^2 - c} $$

where c is a constant (the specific value is not provided but can be any real number). The key features to analyze are:

- Denominator restrictions: Since division by zero is undefined, the denominator cannot be zero:

$$ x^2 - c \neq 0 \Rightarrow x^2 \neq c \Rightarrow x \neq \pm \sqrt{c} $$

- Behavior at critical points: The function approaches infinity or negative infinity as x approaches $\pm\sqrt{c}$, indicating vertical asymptotes at these points.

- Domain of F(x): All real numbers except $x = \pm\sqrt{c}$.

Domain of F(x):

$$ \boxed{\text{All real } x \text{ such that } x \neq \pm \sqrt{c}} $$

Range of F(x):

Since $F(x) = \frac{9}{x^2 - c}$, and as x^2 varies over $[0, \infty)$, the denominator $x^2 - c$ varies over $(-c, \infty)$ (excluding zero). The numerator is positive (9), so:

- When $x^2 - c \to 0^+$, $F(x) \to +\infty$.
- When $x^2 - c \to 0^-$, $F(x) \to -\infty$.
- As $x^2 \to \infty$, $F(x) \to 0^+$.

Therefore, the range depends on the value of c.

Determining the Range of F(x) Based on c

The behavior of F(x) is influenced by whether c is positive, negative, or zero.

Case 1: $c > 0$

- The vertical asymptotes are at $x = \pm \sqrt{c}$.
- For $x^2 > c$, denominator $x^2 - c$ is positive, so $F(x) > 0$.
- For $x^2 < c$, the denominator is negative, so $F(x) < 0$.

In particular:

- As $x \to \pm \sqrt{c}^-$, $F(x) \to -\infty$.
- As $x \to \pm \sqrt{c}^+$, $F(x) \to +\infty$.
- As $x \to \pm \infty$, $F(x) \to 0^+$.

Range of $F(x)$ for $c > 0$:

$$ (-\infty, 0) \cup (0, \infty) $$

excluding zero because $F(x)$ approaches zero but never reaches it.

Case 2: $c < 0$

- Vertical asymptotes at $x = \pm \sqrt{c}$, but since c is negative, $\sqrt{c}$ is imaginary, so no real vertical asymptotes.
- The denominator $x^2 - c$ is always positive because:

$$ x^2 - c = x^2 + |c| > 0 $$

for all real x.

- As $x \to \pm \infty$, $F(x) \to 0^+$.

- The minimum of $x^2 - c$ is at $x=0$, where:

$$ F(0) = \frac{9}{-c} $$

Since $c < 0$, $-c > 0$, so $F(0) > 0$.

- The maximum of $F(x)$ occurs at $x=0$:

$$ F(0) = \frac{9}{-c} $$

- As $x \to \pm \infty$, $F(x) \to 0^+$.

Range of $F(x)$ for $c < 0$:

$$ (0, \frac{9}{-c}) $$

Determining the Domain of the Composite Function (g ∘ F)(x)

To find the domain of $(g \circ F)(x)$, we must consider:

1. The domain of F(x): all real x except where the denominator is zero.
2. The domain of g(y): all y in the range of F(x) for which g(y) is defined.

Suppose g(y) is defined for all real y, or at least for a subset of y-values.

Step-by-step process:

- Step 1: Identify the domain of F(x):

- For $c > 0$: all real x except $\pm \sqrt{c}$.
- For $c < 0$: all real x.

- Step 2: Determine the range of F(x), as discussed in the previous section.

- Step 3: Find the set of y-values where g(y) is defined, which is the intersection of g's domain with the range of F(x).

- Step 4: The domain of $(g \circ F)(x)$ is the set of all x in the domain of F(x) such that F(x) is in g's domain.

This process often involves solving inequalities or restrictions based on g(y).

Analyzing the Range of the Composite Function (g ∘ F)(x)

The range of the composite depends on both the range of F(x) and the behavior of g(y).

Key considerations:

- If g(y) is defined for all real y, then the range of $(g \circ F)(x)$ is simply g applied to the range of F(x).

- If g(y) has restrictions (e.g., g(y) is defined only for $y > 0$), then the range of $(g \circ F)(x)$ is the image of the intersection between the range of F(x) and the domain of g(y).

Practical Examples and Applications

Example 1: g(y) = y^2

Suppose \(g(y) = y^2 \), which is defined for all real y.

- For c > 0:

- Range of F(x): \((-\infty, 0) \cup (0, \infty) \).
- Applying g(y) = y^2:

- The image of F(x) under g is:

\[\{ y^2 \mid y \neq 0 \} = [0, \infty) \]

- The domain of \((g ◦ F)(x) \) is all x where F(x) ≠ 0, which is all real x except \(\pm \sqrt{c} \).

- The range of \((g ◦ F)(x) \):

\[[0, \infty) \]

- For c < 0:

- Range of F(x): \((0, \frac{9}{-c}) \).
- Applying g(y) = y^2:

- The range of \((g ◦ F)(x) \):

\[(0, (\frac{9}{-c})^2) \]

---

0 \), then the points \(x = \pm \sqrt{c} \) are excluded.
- If \(c < 0 \), then \(x^2 \) is always non-negative, so \(x^2 - c > 0 \), making the denominator always positive, and F(x) is defined for all real x.

Summary:

- For \(c > 0 \):

\[
\text{Domain of } F(x): \quad (-\infty, -\sqrt{c}) \cup (-\sqrt{c}, \sqrt{c}) \cup (\sqrt{c}, \infty)
\]

- For \(c \leq 0 \):

\[
\text{Domain of } F(x): \quad \mathbb{R}
\]

Step 2: Determine the Domain of $(g \circ F)(x)$

Since:

\[
(g \circ F)(x) = g(F(x))
\]

and assuming \(g(x) = \sqrt{x} \), the composite exists only where:

1. \(F(x) \) is in the domain of g, i.e.,

\[
F(x) \geq 0
\]

2. \(x \) is in the domain of F, i.e., the points where the denominator is non-zero.

Therefore:

\[
\boxed{
\text{Domain of } (g \circ F)(x) = \left\{ x \in \text{Domain of } F : F(x) \geq 0 \right\}
}
\]

Analyzing \(F(x) = \frac{9}{x^2 - c} \) for \(F(x) \geq 0 \):

Since numerator 9 is positive, the sign of \(F(x) \) depends entirely on the denominator:

\[
x^2 - c
\]

- If \(x^2 - c > 0 \), then \(F(x) > 0 \).
- If \(x^2 - c < 0 \), then \(F(x) < 0 \).
- When \(x^2 - c \to 0 \), \(F(x) \to \pm \infty \).

Thus:

$$
F(x) \geq 0 \quad \Rightarrow \quad x^2 - c > 0
$$

since at points where $x^2 - c = 0$, $F(x)$ is undefined.

Final Domain of the Composite Function

- For $c > 0$:

$$
x^2 - c > 0 \Rightarrow x^2 > c \Rightarrow x < -\sqrt{c} \quad \text{or} \quad x > \sqrt{c}
$$

and from earlier, $F(x)$ is defined on:

$$
(-\infty, -\sqrt{c}) \cup (\sqrt{c}, \infty)
$$

- Since $F(x) \geq 0$ on these intervals, and $g(x) = \sqrt{x}$ is defined for $x \geq 0$, the domain of the composite becomes:

$$
\boxed{
\text{Domain of } (g \circ F)(x) = (-\infty, -\sqrt{c}) \cup (\sqrt{c}, \infty)
}
$$

- For $c \leq 0$, $F(x)$ is always positive (or at least non-negative), so:

$$
\text{Domain of } (g \circ F)(x) = \mathbb{R} \setminus \text{points where } x^2 = c
$$

Determining the Range of $(g \circ F)(x)$

Step 1: Find the Range of F(x)

Given $F(x) = \frac{9}{x^2 - c}$:

- As $x \to \pm \infty$, $x^2 \to \infty$:

$$
F(x) \to 0^+
$$

- Near the vertical asymptotes $x = \pm \sqrt{c}$:

- When approaching from the left or right, $x^2 - c \to 0^+$, so

$$
F(x) \to +\infty
$$

- The behavior of $F(x)$ indicates its range depends on the domain interval.

For $c > 0$:

- On $(-\infty, -\sqrt{c})$:

$x^2 - c > 0$, so $F(x) > 0$, decreasing from $+\infty$ (near $x \to -\sqrt{c}^-$) to 0 (as $x \to -\infty$).

- On $(\sqrt{c}, \infty)$:

Similar behavior applies.

- Therefore, the range of F over these intervals is:

$$
(0, +\infty)
$$

For $c \leq 0$:

- $x^2 - c$ is always positive (or zero at specific points), so $F(x)$ is always positive except at points where denominator is zero.

- The overall range remains $(0, +\infty)$.

Step 2: Find the Range of (g ∘ F)(x)

Given $g(x) = \sqrt{x}$, and $F(x) \in (0, \infty)$, then:

\[
(g \circ F)(x) = \sqrt{F(x)}
\]

- Since \(F(x) > 0 \), \(\sqrt{F(x)} \) ranges from:

\[
0^+ \quad \text{to} \quad +\infty
\]

- The minimum value approaches 0 but never reaches it because \(F(x) \) approaches 0 asymptotically.

Therefore:

\[
\boxed{
\text{Range of } (g \circ F)(x) = (0, +\infty)
}
\]

Summary and Practical Insights

Pros:

- The methodical approach to analyzing the domain and range provides clarity on how the composition behaves.
- Understanding the individual properties of \(F(x) \) and \(g(x) \) allows for precise determination of the combined function's characteristics.
- The process illustrates how asymptotes, positivity, and domain restrictions influence the overall function behavior.

Cons / Challenges:

- The analysis depends

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