Determine The Equation Of A Line Passing Through (3, 2) That Minimizes The Area Bounded By The Line,

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In the realm of analytical geometry, understanding how to determine the optimal line passing through a specific point—such as (3, 2)—that minimizes or maximizes a particular geometric quantity is a fundamental problem. One such intriguing challenge is identifying the equation of a line passing through a given point which minimizes the area enclosed or bounded by the line and other constraints. This problem has wide-ranging applications, from designing efficient enclosures in architecture to optimizing boundary conditions in physics and engineering.

This article delves into the mathematical principles and methods used to find the equation of a line passing through the point (3, 2) that minimizes the area bounded by the line, often in conjunction with other geometric figures or constraints. Our goal is to provide a comprehensive, step-by-step explanation that combines theoretical insights with practical methods, ensuring clarity and relevance for students, educators, and professionals seeking to enhance their understanding of this classical problem.

Understanding the Problem Context

Before embarking on the solution, it is essential to clarify the problem's scope and context.

What does "minimizes the area bounded by the line" mean?
In geometric optimization problems, the phrase typically refers to selecting a line that, together with other given elements (like coordinate axes or other lines), encloses the smallest possible area. For example, this could involve a line passing through a fixed point and intersecting axes or other lines, with the goal of minimizing the resulting enclosed region.

Common Scenario:
Suppose we are asked to find the equation of a line passing through (3, 2) that, when combined with the axes or certain boundary lines, forms a bounded region with the least possible area. For instance, the problem might involve:

- The line passing through (3, 2) and intersecting the coordinate axes.
- The region bounded by this line and the axes or other lines.

- The goal to minimize the area of this bounded region.

This type of problem often appears in calculus and optimization contexts, requiring the formulation of the area as a function of the line's parameters and then finding the minimum.

Mathematical Foundations and Approach

To solve such a problem, we need to:

1. Express the line's equation in a general form passing through the point (3, 2).
2. Identify the bounded region created by the line and the coordinate axes or other constraints.
3. Derive an expression for the area of this bounded region as a function of the line's parameters.
4. Use calculus or algebraic methods to find the minimum of this area function.

Let's proceed step-by-step.

Step 1: General Equation of the Line Passing Through (3, 2)

Any line passing through the point (3, 2) can be expressed in various forms. The slope-intercept form is convenient:

$$ y = m x + c $$

where:
- m is the slope,
- c is the y-intercept.

Since the line passes through (3, 2), it satisfies:

$$ 2 = m \times 3 + c \Rightarrow c = 2 - 3m $$

Thus, the line's equation becomes:

$$ y = m x + (2 - 3m) $$

Alternatively, the point-slope form:

$$ y - 2 = m (x - 3) $$

Step 2: Determining the Bounded Region

To minimize the area, we need to define the bounded region formed by the line and other boundary lines. A common setup involves the coordinate axes:

- The x-axis: $y = 0$
- The y-axis: $x = 0$

Assuming the line intersects the axes, it creates a triangular region with the axes.

Find the intercepts:

- x-intercept: set $y=0$:

$$ 0 = m x + c \Rightarrow x = -\frac{c}{m} = -\frac{2 - 3m}{m} $$

- y-intercept: set $x=0$:

$$ y = c = 2 - 3m $$

Note: For the bounded region to exist in the first quadrant (where $x \geq 0, y \geq 0$), the intercepts should be positive, which imposes constraints on m.

Step 3: Expressing the Area of the Bounded Region

Assuming the line intersects the axes at positive points, the bounded area is a triangle with vertices at:

- The origin $(0, 0)$,
- The x-intercept $\left(-\frac{2 - 3m}{m}, 0\right)$,
- The y-intercept $(0, 2 - 3m)$.

Note: For these intercepts to be in the first quadrant:

- x-intercept > 0:

$$ -\frac{2 - 3m}{m} > 0 $$

- y-intercept > 0:

$$ 2 - 3m > 0 \Rightarrow m < \frac{2}{3} $$

Now, the area A of the triangle formed by these intercepts is:

$$ A = \frac{1}{2} \times (\text{x-intercept}) \times (\text{y-intercept}) $$

Substituting:

$$ A(m) = \frac{1}{2} \times \left(-\frac{2 - 3m}{m}\right) \times (2 - 3m) $$

Simplify:

$$ A(m) = \frac{1}{2} \times \left(-\frac{2 - 3m}{m}\right) \times (2 - 3m) $$

$$ A(m) = -\frac{1}{2} \times \frac{(2 - 3m)^2}{m} $$

Given the negative sign, and considering the geometric interpretation (area must be positive), we focus on the absolute value:

$$ A(m) = \frac{1}{2} \times \frac{(2 - 3m)^2}{|m|} $$

Since $m < 0$ for the intercepts to be positive, we can write:

$$ A(m) = \frac{1}{2} \times \frac{(2 - 3m)^2}{-m} $$

for $m < 0$.

Step 4: Minimizing the Area Function

Our goal is now to find the value of m that minimizes $A(m)$:

$$ A(m) = \frac{(2 - 3m)^2}{-2m} $$

for $m < 0$.

Let's define:

$$ A(m) = \frac{(2 - 3m)^2}{-2m} $$

Simplify numerator:

$$ (2 - 3m)^2 = 4 - 12m + 9m^2 $$

Thus:

$$ A(m) = \frac{4 - 12m + 9m^2}{-2m} $$

Now, to find the minimum, differentiate $A(m)$ with respect to m, set the derivative equal to zero, and solve for m.

Calculus-Based Optimization

Differentiate $A(m)$:

Express $A(m)$ as:

$$ A(m) = \frac{4 - 12m + 9m^2}{-2m} $$

Rewrite as:

$$ A(m) = -\frac{1}{2m} \times (4 - 12m + 9m^2) $$

Let:

$$ A(m) = f(m) = -\frac{1}{2m} \times g(m) $$

where $g(m) = 4 - 12m + 9m^2$.

Apply the quotient rule or product rule accordingly.

Alternatively, rewrite $A(m)$ as:

$$ A(m) = -\frac{1}{2m} \times (4 - 12m + 9m^2) $$

Differentiate:

$$ A'(m) = \text{derivative of numerator} \times \text{denominator} - \text{numerator} \times \text{derivative of denominator} \div (\text{denominator})^2 $$

But perhaps easier to write $A(m)$ explicitly:

$$ A(m) = \frac{4 - 12m + 9m^2}{-2m} $$

which simplifies to:

$$ A(m) = -\frac{4 - 12m + 9m^2}{2m} $$

Now, differentiate:

$$ A'(m) = - \frac{d}{dm} \left(\frac{4 - 12m + 9m^2}{2m}\right) $$

Let’s define:

$$ Q(m) = \frac{4 - 12m + 9m^2}{2m} $$

then:

$$ A(m) = - Q(m) $$

Find

Frequently Asked Questions

How do I find the equation of a line passing through (3, 2) that minimizes the area bounded by the line?

To find such a line, you need to set up the line's equation passing through (3, 2), express the area bounded by the line and the axes, and then minimize this area using calculus or geometric reasoning.

What is the general form of the line passing through point (3, 2)?

The general form is $y - 2 = m(x - 3)$, or $y = m(x - 3) + 2$, where m is the slope of the line.

How is the area bounded by the line and the axes related to the slope m?

The bounded area is a triangle with vertices at the intercepts of the line with the axes, which depend on the slope m. Expressing these intercepts allows us to formulate the area as a function of m.

How do I find the intercepts of the line in terms of the slope m?

The x-intercept occurs where $y=0$: $0 = m(x - 3) + 2$, solving for x gives $x = 3 - 2/m$. The y-intercept occurs where $x=0$: $y = m(0 - 3) + 2 = -3m + 2$.

What is the formula for the area bounded by the line and the axes?

The area of the triangle is (1/2) |x-intercept| |y-intercept|, which becomes $(1/2)\ |(3 - 2/m)|\ |(-3m + 2)|$ in terms of m.

How do I minimize the area function with respect to the slope m?

Differentiate the area function with respect to m, set the derivative equal to zero, and solve for m to find the slope that minimizes the area.

What constraints are there on the slope m when minimizing the area?

The slope m must be chosen so that the intercepts are positive or within the relevant domain to form a bounded triangle, and avoid division by zero or undefined expressions.

What is the resulting equation of the line after minimizing the area?

After finding the optimal slope m, substitute it back into $y = m(x - 3) + 2$ to get the equation of the line passing through (3, 2) that minimizes the bounded area.

Can you provide a step-by-step example of solving for the minimal area line?

Yes. First, express the intercepts in terms of m, then write the area as a function of m, differentiate to find critical points, solve for m, and finally write the line's equation with that slope.

Why is minimizing the area bounded by the line and axes important, and how is it applied?

Minimizing the bounded area helps in optimization problems related to design, resource allocation, or geometric constraints, and involves calculus techniques to find the most efficient or optimal line configuration.

Additional Resources

Determine The Equation Of A Line Passing Through (3, 2) That Minimizes The Area Bounded By The Line

When exploring the intriguing problem of determine the equation of a line passing through (3, 2) that minimizes the area bounded by the line, we delve into a classic optimization challenge rooted in coordinate geometry and calculus. This problem not only sharpens our analytical skills but also offers insight into how geometric configurations and algebraic expressions intertwine to produce minimal or optimal solutions. Whether you're a student tackling calculus problems or a professional interested in geometric optimization, understanding how to approach this problem step-by-step is invaluable.

Understanding the Problem

Before jumping into the solution, it's essential to clarify what the problem entails:

- Given point: (3, 2)
- Objective: Find the equation of a line passing through this point that minimizes the area of a certain bounded region.
- Question: What is the minimal area, and what is the equation of the line that achieves this?

The challenge lies in defining the bounded region. Typically, in such problems, the line intersects the coordinate axes, forming a bounded triangle with the axes, and the goal is to minimize this triangle's area.

Visualizing the Geometric Setup

Imagine plotting a line on the Cartesian plane passing through the point (3, 2). For each possible line passing through this point, the line will intersect the axes at some points, say:

- x-intercept: point where the line crosses the x-axis.
- y-intercept: point where the line crosses the y-axis.

These intercepts define a triangle with the axes, and the area of this triangle can be expressed as a function of the line's parameters. To minimize this area, we need to:

1. Express the line's equation in a general form passing through (3, 2).
2. Find the intercepts in terms of the line's parameters.
3. Express the area of the bounded triangle in terms of these parameters.
4. Use calculus techniques to find the minimum of this area function.
5. Determine the specific line equation that yields this minimum.

Step 1: Expressing the Equation of the Line

Any line passing through (3, 2) can be written using the point-slope form:

$$ y - 2 = m (x - 3) $$

where m is the slope of the line, which can vary.

Alternatively, this can be rearranged into slope-intercept form:

$$ y = m x - 3 m + 2 $$

or in standard form:

$$ y = m x + (2 - 3 m) $$

Step 2: Finding Intercepts in Terms of the Slope

x-intercept:

Set $y = 0$:

$$ 0 = m x + (2 - 3 m) $$

$$ m x = - (2 - 3 m) $$

$$ x = - \frac{2 - 3 m}{m} $$

$$ x = - \frac{2}{m} + 3 $$

y-intercept:

Set $x = 0$:

$$ y = m \times 0 + (2 - 3 m) = 2 - 3 m $$

Step 3: Expressing the Area of the Bounded Triangle

The triangle bounded by the coordinate axes and the line has vertices at:

- The origin (0, 0)
- The x-intercept: $\left(-\frac{2}{m} + 3, 0 \right)$
- The y-intercept: $(0, 2 - 3 m)$

Since the line passes through (3, 2), and the intercepts depend on m, the area of this triangle can be calculated as:

$$ \text{Area} = \frac{1}{2} \times |\text{x-intercept}| \times |\text{y-intercept}| $$

Note: For the area to be positive and bounded in the usual sense, the intercepts should be positive, but the absolute value ensures the area is positive regardless of the line's slope.

Expressed explicitly:

$$ A(m) = \frac{1}{2} \left| - \frac{2}{m} + 3 \right| \times |2 - 3 m| $$

Step 4: Simplify the Area Function

To proceed with optimization, it’s often easier to work with the non-absolute value form, considering the appropriate domains where the intercepts are positive or negative.

Let’s analyze the signs:

- x-intercept: $- \frac{2}{m} + 3$
- y-intercept: $2 - 3 m$

Assuming both intercepts are positive for simplicity:

$$ - \frac{2}{m} + 3 > 0 \Rightarrow - \frac{2}{m} > -3 \Rightarrow \frac{2}{m} < 3 $$

Similarly, $2 - 3 m > 0 \Rightarrow m < \frac{2}{3}$

To keep the analysis straightforward, consider m in the domain $m < \frac{2}{3}$ and $m \neq 0$.

Expressing the area without absolute values:

$$ A(m) = \frac{1}{2} \left(3 - \frac{2}{m} \right) (2 - 3 m) $$

Step 5: Simplify and Derive the Area Function

Multiply out numerator and denominator:

$$ A(m) = \frac{1}{2} \left(3 - \frac{2}{m} \right) (2 - 3 m) $$

Express $3 - \frac{2}{m}$ as a single fraction:

$$ 3 - \frac{2}{m} = \frac{3 m - 2}{m} $$

Now, rewrite the area:

$$ A(m) = \frac{1}{2} \times \frac{3 m - 2}{m} \times (2 - 3 m) $$

Multiply numerator terms:

$$ (3 m - 2)(2 - 3 m) $$

Expand:

$$ 3 m \times 2 = 6 m $$
$$ 3 m \times (-3 m) = -9 m^2 $$
$$ -2 \times 2 = -4 $$
$$ -2 \times (-3 m) = 6 m $$

Sum these:

$$ 6 m - 9 m^2 - 4 + 6 m = (6 m + 6 m) - 9 m^2 - 4 = 12 m - 9 m^2 - 4 $$

So, the area becomes:

$$ A(m) = \frac{1}{2} \times \frac{12 m - 9 m^2 - 4}{m} $$

Express as:

$$ A(m) = \frac{12 m - 9 m^2 - 4}{2 m} $$

Step 6: Find the Value of m That Minimizes $A(m)$

To find the minimum, differentiate $A(m)$ with respect to m, set derivative to zero, and solve for m.

Express $A(m)$:

$$ A(m) = \frac{12 m - 9 m^2 - 4}{2 m} $$

Rewrite numerator:

$$ N(m) = 12 m - 9 m^2 - 4 $$

Denominator:

$$ D = 2 m $$

Therefore:

$$ A(m) = \frac{N(m)}{D} $$

Apply the quotient rule:

$$ A'(m) = \frac{N'(m) \times D - N(m) \times D'}{D^2} $$

Calculate derivatives:

- $N'(m) = 12 - 18 m$
- $D' = 2$

Plug into the quotient rule:

\[A'(m) = \frac{(12 - 18 m) \times 2 m - (12 m - 9 m^2 - 4) \times 2}{(2 m)^2} \]

Simplify numerator:

\[(12 - 18 m) \times 2 m = 24 m - 36 m^2 \]

\[(12 m - 9 m^2 - 4) \times 2 = 24 m - 18 m^2 - 8 \]

Now, numerator:

\[24 m - 36 m^2 - (24 m - 18 m^2 - 8) = 24 m - 36 m^2 - 24 m + 18 m^2 + 8 \]

Simplify:

\[(24 m - 24 m) + (-36 m^2 + 18 m^2) + 8 = 0 - 18 m^2 + 8 \]

Thus:

\[A'(m) = \frac{-18 m^2 + 8}{(2 m)^2} \]

Set numerator to zero to find critical points:

\[-18 m^2 + 8 = 0 \]

\[18 m^2 = 8 \]

\[m^2 = \frac{8}{18}

Determine The Equation Of A Line Passing Through 3 2 That Minimizes The Area Bounded By The Line

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