Determine The Extreme Point (x*, Y*) And Its Nature Of The Following Function: Z = 25x + 16y - 4xy -

Determine The Extreme Point (x, Y) And Its Nature Of The Following Function: Z = 25x + 16y - 4xy -

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Introduction

In the fields of mathematics, economics, engineering, and optimization, understanding the behavior of functions is crucial. One of the fundamental tasks is identifying the extreme points—points where the function reaches its maximum or minimum values. This process involves analyzing the function's critical points and determining their nature—whether they are maxima, minima, or saddle points.

In this article, we focus on the function:

$$Z = 25x + 16y - 4xy$$

Our goal is to determine the extreme point(s) (specifically, the coordinates \(x^\) and \(Y^\)) and analyze the nature of these points. This involves finding the critical points through calculus techniques, such as partial derivatives, and then applying second derivative tests to classify these points.

Understanding the Function

The Function $Z = 25x + 16y - 4xy$

This is a bilinear function involving two variables, x and y. It combines linear terms and a product term, which makes it a quadratic form with respect to the variables. Recognizing the structure of the function is essential for applying calculus techniques for optimization.

Key Concepts

Before proceeding, it's important to understand some foundational concepts:

- Critical Point: A point (x, y) where the partial derivatives of the function are zero (or undefined). These are potential candidates for maxima, minima, or saddle points.
- Second Derivative Test: A method to classify critical points based on second derivatives.

Step-by-Step Process to Find the Extreme Point(s)

Step 1: Compute the First-Order Partial Derivatives

To locate critical points, we need the partial derivatives of Z with respect to x and y:

$$\frac{\partial Z}{\partial x} \quad \text{and} \quad \frac{\partial Z}{\partial y}$$

Calculating these derivatives:

$$\frac{\partial Z}{\partial x} = 25 - 4y$$

$$
\frac{\partial Z}{\partial y} = 16 - 4x
$$

Step 2: Set Partial Derivatives to Zero

To find critical points, solve the system:

$$
\frac{\partial Z}{\partial x} = 0 \Rightarrow 25 - 4y = 0
$$
$$
\frac{\partial Z}{\partial y} = 0 \Rightarrow 16 - 4x = 0
$$

Solving these equations:

$$
25 - 4y = 0 \Rightarrow y = \frac{25}{4} = 6.25
$$
$$
16 - 4x = 0 \Rightarrow x = \frac{16}{4} = 4
$$

Critical Point:

$$
(x^{\wedge}, y^{\wedge}) = (4, 6.25)
$$

This is the potential extremum.

Analyzing the Nature of the Critical Point

Step 3: Compute the Second-Order Partial Derivatives

To classify the critical point, calculate the second derivatives:

$$
\frac{\partial^2 Z}{\partial x^2} = 0
$$
$$
\frac{\partial^2 Z}{\partial y^2} = 0
$$
$$
\frac{\partial^2 Z}{\partial x \partial y} = \frac{\partial^2 Z}{\partial y \partial x} = -4
$$

Step 4: Apply the Second Derivative Test

The second derivative test uses the Hessian determinant:

$$
D = \left(\frac{\partial^2 Z}{\partial x^2} \right) \left(\frac{\partial^2 Z}{\partial y^2} \right) - \left(\frac{\partial^2 Z}{\partial x \partial y} \right)^2
$$

Plugging in the values:

$$
D = (0)(0) - (-4)^2 = 0 - 16 = -16
$$

Step 5: Interpret the Result

- If $D > 0$ and $\frac{\partial^2 Z}{\partial x^2} > 0$, then $(x^\wedge, y^\wedge)$ is a local minimum.
- If $D > 0$ and $\frac{\partial^2 Z}{\partial x^2} < 0$, then $(x^\wedge, y^\wedge)$ is a local maximum.
- If $D < 0$, the point is a saddle point.

Since $D = -16 < 0$, the critical point $(4, 6.25)$ is a saddle point.

Implications of the Saddle Point

The saddle point indicates that the function Z does not have a local maximum or minimum at $(4, 6.25)$. Instead, the surface of the function curves upward in some directions and downward in others, which is characteristic of saddle points.

Additional Analysis: Global Extrema and Practical Significance

Is there a global maximum or minimum?

Given the bilinear nature of the function, it does not have finite global maxima or minima over the entire plane $\mathbb{R}^2$. The function can tend to infinity or negative infinity depending on the direction of approach.

Practical interpretation in real-world applications:

- In economic modeling, such a function might represent profit or cost functions where the saddle point indicates an equilibrium or a point of instability.
- In optimization problems, recognizing the saddle point helps in understanding that the critical point is not an optimal point but a point of inflection.

Visualizing the Function and Its Critical Point

Graphical Representation

A 3D surface plot of $Z = 25x + 16y - 4xy$ would show a saddle surface, with the critical point at $(4, 6.25)$. The surface would rise in some directions and fall in others, illustrating the saddle nature.

Contour Plot

Contour plots depict lines of constant Z. Near the saddle point, contours would form hyperbolas, further illustrating the saddle surface.

Summary of Findings

- Critical Point: $(x^\wedge, y^\wedge) = (4, 6.25)$
- Nature of Critical Point: Saddle point (neither a maximum nor a minimum)
- Second Derivative Test Result: $D = -16 < 0$

Conclusion

The detailed analysis of the function \(Z = 25x + 16y - 4xy \) reveals a critical point at \((4, 6.25)\), which is classified as a saddle point based on second derivative tests. Understanding the nature of such points is essential in optimization, especially when deploying these functions in real-world scenarios like economics, engineering, or data science.

By systematically calculating the partial derivatives, solving for critical points, and applying the second derivative test, we gain valuable insights into the behavior of the function. Recognizing saddle points helps avoid misinterpretations of local maxima or minima and guides decision-making processes in complex systems.

Additional Tips for Analyzing Similar Functions

- Always verify the critical points by solving the system of partial derivatives.
- Use the Hessian determinant to classify critical points; remember the sign conventions.
- For functions with multiple variables, consider plotting the surface or contour maps for better visualization.
- Beware of unbounded functions where global extrema may not exist.

SEO Keywords for Optimization

- Extreme points of functions
- Critical points and their classification
- Second derivative test for multivariable functions
- Saddle points in calculus
- Optimization of bilinear functions
- Analyzing functions with partial derivatives
- Mathematical analysis of functions with two variables
- How to find maxima and minima in multivariable calculus

Final Thoughts

Mastering the process of determining extreme points and understanding their nature is fundamental for anyone involved in mathematical modeling, optimization, or applied sciences. The case of the function \(Z = 25x + 16y - 4xy \) exemplifies the importance of calculus tools in analyzing complex functions, enabling informed decisions and accurate interpretations in practical applications.

Frequently Asked Questions

How do you find the extreme point (x, y) of the function Z = 25x + 16y - 4xy?

To find the extreme point, compute the partial derivatives of Z with respect to x and y, set them equal to zero, and solve the resulting system of equations for x and y.

What are the steps to determine the nature (maximum, minimum, saddle point) of the extreme point for the given function?

After finding the critical point(s), evaluate the second derivatives to compute the Hessian matrix. The sign of its determinant determines the nature: positive indicates a local minimum or maximum, negative indicates a saddle point.

What are the partial derivatives of Z = 25x + 16y - 4xy?

The partial derivatives are: $\partial Z/\partial x = 25 - 4y$ and $\partial Z/\partial y = 16 - 4x$.

How do you solve for the critical point (x, y) from the partial derivatives?

Set the partial derivatives equal to zero: $25 - 4y = 0$ and $16 - 4x = 0$, then solve simultaneously for x and y.

What is the critical point (x, y) for the function Z = 25x + 16y - 4xy?

Solving $25 - 4y = 0$ gives $y = 25/4 = 6.25$; solving $16 - 4x = 0$ gives $x = 4$.

How do you determine the nature of the critical point (x, y) using second derivatives?

Calculate the second derivatives: $\partial^2 Z/\partial x^2 = 0$, $\partial^2 Z/\partial y^2 = 0$, and $\partial^2 Z/\partial x \partial y = -4$. Then evaluate the Hessian determinant $D = (\partial^2 Z/\partial x^2)(\partial^2 Z/\partial y^2) - (\partial^2 Z/\partial x \partial y)^2$.

What is the value of the Hessian determinant for this function at the critical point?

Since $\partial^2 Z/\partial x^2 = 0$ and $\partial^2 Z/\partial y^2 = 0$, and $\partial^2 Z/\partial x \partial y = -4$, $D = (0)(0) - (-4)^2 = -16$.

Based on the Hessian determinant, what is the nature of the critical point for Z = 25x + 16y - 4xy?

Since $D = -16 < 0$, the critical point is a saddle point.

What is the significance of identifying the saddle point in this function?

Identifying a saddle point indicates that the critical point is neither a maximum nor a minimum; the function behaves differently in different directions around this point, which is important for optimization and analysis.

Can the critical point (x, y) = (4, 25/4) be a global maximum or minimum for the function?

No, because the Hessian determinant indicates a saddle point, which is neither a global maximum nor a minimum.

Additional Resources

Extreme Point Determination of the Function Z = 25x + 16y - 4xy: An In-Depth Analysis

In the realm of mathematical optimization, understanding the nature and location of extreme points—maxima, minima, or saddle points—is fundamental. These points often represent optimal solutions in various fields such as economics, engineering, and operations research. Today, we dissect a compelling function:

Z = 25x + 16y - 4xy

Our goal is to determine its extreme points (x, y) and analyze their nature—whether they are maxima, minima, or saddle points. This in-depth exploration is designed to give you a comprehensive understanding of the mathematical process involved, akin to a detailed expert review.

Understanding the Function and Its Context

The function Z = 25x + 16y - 4xy is a bivariate quadratic form, linear in each variable but with a mixed term (-4xy). Such functions often model real-world scenarios involving two interdependent variables—for example, cost functions, profit maximization, or resource allocation problems.

Key features of this function:
- Linear components: 25x and 16y suggest a linear growth in each variable.
- Interaction term (-4xy): introduces a coupling effect, which can complicate the analysis by creating curvature and potential saddle points.

This function's structure hints at the possibility of a saddle point, maximum, or minimum depending on the combined behavior of its derivatives and second derivatives.

Step 1: Finding Critical Points (x, y)

Critical points are locations where the first derivatives of Z vanish—that is, where the function has potential maxima, minima, or saddle points.

Calculating the first derivatives:

$$
\frac{\partial Z}{\partial x} = 25 - 4y
$$
$$
\frac{\partial Z}{\partial y} = 16 - 4x
$$

Setting these derivatives to zero:

$$
25 - 4y = 0 \Rightarrow y = \frac{25}{4} = 6.25
$$
$$
16 - 4x = 0 \Rightarrow x = \frac{16}{4} = 4
$$

Critical point:

\[
(x^, y^) = (4, 6.25)
\]

Interpretation: At (4, 6.25), the function's slope in both directions is zero, making it a candidate for an extremum.

Step 2: Determining the Nature of the Critical Point

Finding the critical point is only part of the analysis. To classify whether this point is a maximum, minimum, or saddle point, we examine the second derivatives and apply the second derivative test for functions of two variables.

Calculating second derivatives:

$$
\frac{\partial^2 z}{\partial x^2} = 0
$$
$$
\frac{\partial^2 z}{\partial y^2} = 0
$$
$$
\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = -4
$$

Construct the Hessian matrix:

$$
H = \begin{bmatrix}
\frac{\partial^2 z}{\partial x^2} & \frac{\partial^2 z}{\partial x \partial y} \\
\frac{\partial^2 z}{\partial y \partial x} & \frac{\partial^2 z}{\partial y^2}
\end{bmatrix} =
\begin{bmatrix}
0 & -4 \\
-4 & 0
\end{bmatrix}
$$

Applying the second derivative test:

- Compute the determinant of the Hessian:

$$
D = \left(\frac{\partial^2 z}{\partial x^2} \right) \left(\frac{\partial^2 z}{\partial y^2} \right) - \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 = (0)(0) - (-4)^2 = -16
$$

- Interpretation:
- If $D > 0$ and $\frac{\partial^2 z}{\partial x^2} > 0$, then the point is a local minimum.
- If $D > 0$ and $\frac{\partial^2 z}{\partial x^2} < 0$, then the point is a local maximum.
- If $D < 0$, then the point is a saddle point.

Since $D = -16 < 0$, the critical point at (4, 6.25) is a saddle point.

Implications of the Saddle Point

The negative determinant indicates this critical point is a saddle point—a point where the function curves upward in one direction and downward in another. Such points are neither maxima nor minima but are crucial in understanding the topography of the function.

Practical insight:
- In optimization problems, saddle points often indicate regions of unstable equilibrium.
- For practical applications, understanding the saddle point helps in designing constraints or bounds to avoid non-optimal solutions.

Additional Considerations: Domain and Constraints

While the mathematical analysis indicates a saddle point at (4, 6.25), real-world applications often impose constraints on x and y (such as non-negativity, upper bounds, or other restrictions).

If constraints exist:
- The global maximum or minimum might occur at boundary points or within feasible regions.
- The saddle point may not be relevant if it lies outside the domain of interest.

For unconstrained optimization:
- The saddle point remains the only critical point, highlighting the need for further analysis if seeking to optimize z.

Summary of Findings

Aspect	Result	Explanation
Critical point (x, y)	(4, 6.25)	Found by setting partial derivatives to zero.
Second derivative test	Saddle point	Because the Hessian determinant is negative (-16).
Nature of the point	Saddle point	Indicates a point of inflection in the function's surface.

Final Remarks: Practical Relevance and Broader Context

This detailed analysis underscores the importance of second derivative tests in understanding the landscape of complex functions. For practitioners, recognizing saddle points is crucial—they signal regions where the function does not attain a local extremum, which can influence decision-making in optimization routines.

In real-world applications:
- Engineers might use this insight to avoid configurations corresponding to saddle points.
- Economists could interpret saddle points as unstable equilibria in economic models.
- Operations researchers may incorporate domain constraints to shift the focus towards boundary optima rather than saddle points.

In conclusion:

The function $Z = 25x + 16y - 4xy$ exhibits a saddle point at (4, 6.25), with no local maxima or minima within the unconstrained domain. This analysis exemplifies the importance of critical point evaluation and second derivative testing in understanding the fundamental nature of multivariable functions.

Expert tip: Always consider the domain constraints and the physical interpretation of variables when applying such analyses to real-world problems. Mathematical critical points provide valuable insights, but the context determines their practical significance.

End of in-depth analysis.

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