Determine The Values Of The Parameter S For Which The System Has A Unique Solution, And Describe The

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Understanding when a system of equations has a unique solution is fundamental in linear algebra and its applications across engineering, physics, economics, and computer science. The parameter S often appears as a variable or a coefficient within the system, influencing the system's solvability. This comprehensive guide aims to help you analyze such systems, determine the specific values of S that guarantee a unique solution, and interpret the implications of these findings.

Introduction to Systems of Linear Equations and Parameters

A system of linear equations consists of multiple equations involving the same set of variables. The general form can be expressed as:

$$
A \mathbf{x} = \mathbf{b}
$$

where:
- A is the coefficient matrix,
- $\mathbf{x}$ is the vector of variables,
- $\mathbf{b}$ is the constant vector.

The inclusion of a parameter S typically appears within the coefficient matrix A or the vector $\mathbf{b}$, transforming the system into a parametric family of systems:

$$
A(S) \mathbf{x} = \mathbf{b}(S)
$$

The goal is to determine the values of S for which the system admits a unique solution, meaning that for those specific values, the system's solution exists and is uniquely determined.

Fundamental Concepts for Analyzing System Solutions

Before delving into the parametrized system, it is essential to understand key concepts:

1. Consistency of the System

- A system is consistent if at least one solution exists.
- It is inconsistent if no solutions exist.

2. Number of Solutions

- Unique solution: exactly one solution.
- Infinite solutions: infinitely many solutions, typically occurring when the system is underdetermined or has dependent equations.

3. Role of the Coefficient Matrix Determinant

- The determinant of the coefficient matrix, $\det(A)$, plays a critical role.
- If $\det(A) \neq 0$, the system has a unique solution.
- If $\det(A) = 0$, the system may have either no solutions or infinitely many solutions.

Analyzing a Parametric System: Step-by-Step Approach

To determine the specific values of S for which the system has a unique solution, follow these steps:

Step 1: Write the System in Matrix Form

Express the system explicitly, noting the dependence on S. For example:

\[
\begin{cases}
a_{11}(S) x_1 + a_{12}(S) x_2 + \dots + a_{1n}(S) x_n = b_1(S) \\
a_{21}(S) x_1 + a_{22}(S) x_2 + \dots + a_{2n}(S) x_n = b_2(S) \\
\vdots \\
a_{m1}(S) x_1 + a_{m2}(S) x_2 + \dots + a_{mn}(S) x_n = b_m(S)

\end{cases}
\]

Step 2: Identify the Coefficient Matrix \(A(S)\)

Construct the matrix:

\[
A(S) = \begin{bmatrix}
a_{11}(S) & a_{12}(S) & \dots & a_{1n}(S) \\
a_{21}(S) & a_{22}(S) & \dots & a_{2n}(S) \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1}(S) & a_{m2}(S) & \dots & a_{mn}(S)
\end{bmatrix}
\]

and the vector:

\[
\mathbf{b}(S) = \begin{bmatrix}
b_1(S) \\
b_2(S) \\
\vdots \\
b_m(S)
\end{bmatrix}
\]

Step 3: Compute \(\det(A(S))\)

Determine the determinant as a function of \(S\). This often involves algebraic expansion or applying properties of determinants for matrices with parametric entries.

Step 4: Find the Values of \(S\) Making \(\det(A(S)) \neq 0\)

- The system has a unique solution whenever \(\det(A(S)) \neq 0\).
- Solve the inequality \(\det(A(S)) \neq 0\) to identify the permissible values of \(S\).

Step 5: Verify the Consistency of the System at These Values

- Even if \(\det(A(S)) \neq 0\), ensure that the system is consistent for these \(S\) by checking if the augmented matrix \([A(S) | \mathbf{b}(S)]\) has the same rank as \(A(S)\).
- Typically, for \(\det(A(S)) \neq 0\), the system is consistent and has a

unique solution.

Practical Examples and Application

Let's analyze a concrete example to understand the process better.

Example System:

\[
\begin{cases}
x + 2y + Sz = 4 \\
2x + (S+1)y + 3z = 7 \\
3x + 4y + (2S)z = 10
\end{cases}
\]

Objective: Find the values of \(S\) for which this system has a unique solution.

Solution Steps:

1. Construct the coefficient matrix:
   \[
   A(S) = \begin{bmatrix}
   1 & 2 & S \\
   2 & S+1 & 3 \\
   3 & 4 & 2S
   \end{bmatrix}
   \]

2. Calculate the determinant:
   \[
   \det(A(S)) = \begin{vmatrix}
   1 & 2 & S \\
   2 & S+1 & 3 \\
   3 & 4 & 2S
   \end{vmatrix}
   \]

3. Compute \(\det(A(S))\):
   \[

$$
\det(A(S)) = 1 \cdot \begin{vmatrix} S+1 & 3 \\ 4 & 2S \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 3 \\ 3 & 2S \end{vmatrix} + S \cdot \begin{vmatrix} 2 & S+1 \\ 3 & 4 \end{vmatrix}
$$

Calculate minors:

$$
\begin{aligned}
& \begin{vmatrix} S+1 & 3 \\ 4 & 2S \end{vmatrix} = (S+1)(2S) - 4 \times 3 = 2S(S+1) - 12 = 2S^2 + 2S - 12 \\
& \begin{vmatrix} 2 & 3 \\ 3 & 2S \end{vmatrix} = 2 \times 2S - 3 \times 3 = 4S - 9 \\
& \begin{vmatrix} 2 & S+1 \\ 3 & 4 \end{vmatrix} = 2 \times 4 - (S+1) \times 3 = 8 - 3S - 3 = 5 - 3S
\end{aligned}
$$

Now substitute back:

$$
\det(A(S)) = 1 \times (2S^2 + 2S - 12) - 2 \times (4S - 9) + S \times (5 - 3S)
$$

Simplify:

$$
= 2S^2 + 2S - 12 - 8S + 18 + 5S - 3S^2
$$

Combine like terms:

$$
(2S^2 - 3S^2) + (2S - 8S + 5S) + (-12 + 18) = -S^2 - 1S + 6
$$

Therefore:

$$
\det(A(S)) = -S^2 - S + 6
$$

1. Find values of S such that $\det(A(S)) \neq 0$:

   $$
   -S^2 - S + 6 \neq 0
   $$

   or equivalently:

   $$
   S^2 + S - 6 \neq 0
   $$

   Factor:

   $$
   (S + 3)(S - 2) \neq 0
   $$

   so:

   $$
   S \neq -3, \quad S \neq 2
   $$

Conclusion:

- The system has a unique solution for all real values of \(S\) except \(S = -3\) and

Frequently Asked Questions

How do I determine the values of the parameter S for which a linear system has a unique solution?

To find the values of S that ensure a system has a unique solution, set up the coefficient matrix and compute its determinant. The system has a unique solution when the determinant is non-zero; solve for S when the determinant $\neq 0$.

What does it mean if the determinant of the coefficient matrix is zero in relation to the parameter S?

If the determinant equals zero for certain values of S, the system either has infinitely many solutions or no solution. A non-zero determinant indicates a unique solution, so the system is only uniquely solvable when the determinant $\neq 0$.

Can you give an example of how to find the values of S for a 2x2 system?

Yes. For a system with coefficient matrix [[a, b], [c, d]], compute the determinant ad - bc. If the entries depend on S, set the determinant equal to zero and solve for S. The values where the determinant $\neq 0$ are the S-values for which the system has a unique solution.

How does the description of the system change

when the parameter S affects the solution set?

When S varies, it can cause the system to transition from having a unique solution to either infinitely many solutions or none. The values of S where the determinant is zero mark these transition points, while non-zero determinant values indicate a unique solution.

Why is it important to identify the parameter values for which a system has a unique solution?

Identifying these values helps in understanding system behavior, ensuring solutions exist and are unique for modeling, analysis, or problem-solving purposes, especially when parameters influence the system's solvability.

What are common methods to describe the solution set for different values of S?

Common methods include analyzing the determinant of the coefficient matrix, using parametric equations for the solution set when free variables exist, and describing the conditions under which the system is consistent and uniquely solvable based on S.

Additional Resources

Determine The Values Of The Parameter S For Which The System Has A Unique Solution, And Describe The

In the realm of linear algebra, systems of equations are fundamental tools used across countless disciplines—from engineering and physics to economics and computer science. Among the key questions mathematicians and scientists often pose is: For which values of a parameter—in this case, S—does a given system possess a unique solution? Understanding this not only clarifies the behavior of the system but also guides practical applications, such as designing stable engineering structures or optimizing algorithms.

This article delves into the process of determining the specific values of S that grant a system a unique solution. We will explore the foundational concepts, examine the role of parameters within systems, and illustrate the methods used to analyze and interpret these conditions. Whether you're a student, researcher, or professional, this comprehensive guide aims to shed light on the intricate relationship between parameters and the solvability of linear systems.

Understanding the Fundamental Concepts

What Is a System of Linear Equations?

A system of linear equations comprises multiple equations involving the same set of variables. For example, a two-variable system might look like:

$$
\begin{cases}
a_{11}x + a_{12}y = b_1 \\
a_{21}x + a_{22}y = b_2
\end{cases}
$$

In matrix form, this is represented as:

$$
A\mathbf{x} = \mathbf{b}
$$

where A is the coefficient matrix, $\mathbf{x}$ is the vector of variables, and $\mathbf{b}$ is the constants vector.

The Role of Parameters in Systems

Parameters are variables that influence the coefficients or constants within the system but are not the primary variables we seek to solve for. In our discussion, S acts as such a parameter, affecting the coefficients or constants. Changing the value of S can alter the nature of the solution set—potentially making the system inconsistent, having infinitely many solutions, or a unique solution.

Conditions for a Unique Solution

A system has a unique solution if and only if the coefficient matrix A is invertible. For square systems (where the number of equations equals the number of variables), this condition simplifies to:

- The determinant of A is non-zero.

Mathematically:

$$
\det(A) \neq 0

\]

When the determinant is zero, the system may have infinitely many solutions or none at all, depending on the consistency conditions.

Analyzing the System with Parameter S

Setting Up the System

Suppose we are given a linear system with a parameter S involved in the coefficients:

$$
\begin{cases}
a_{11}(S) x + a_{12}(S) y = b_1(S) \\
a_{21}(S) x + a_{22}(S) y = b_2(S)
\end{cases}
$$

Here, the coefficients $a_{ij}(S)$ and constants $b_i(S)$ depend on S. Our goal is to find all values of S such that the system has a unique solution.

Computing the Determinant

The key step involves calculating the determinant of the coefficient matrix:

$$
A(S) = \begin{bmatrix}
a_{11}(S) & a_{12}(S) \\
a_{21}(S) & a_{22}(S)
\end{bmatrix}
$$

$$
\det(A(S)) = a_{11}(S) \times a_{22}(S) - a_{12}(S) \times a_{21}(S)
$$

The values of S for which $\det(A(S)) \neq 0$ ensure the system has a unique solution. Conversely, when $\det(A(S)) = 0$, the system either has infinitely many solutions or none, depending on the consistency conditions.

Solving for S

The process involves:

1. Expressing the determinant as a function of S.

2. Finding the roots of $\det(A(S)) = 0$.
3. Determining the values of S that make the determinant non-zero—these are the values for which the system has a unique solution.

Case Study: An Illustrative Example

Let's consider a specific example to concretize these ideas.

Suppose the system is:

$$
\begin{cases}
(2S + 1) x + (3S - 2) y = 5 \\
(S - 1) x + (4S + 3) y = 7
\end{cases}
$$

Step 1: Write the coefficient matrix and compute the determinant:

$$
A(S) = \begin{bmatrix}
2S + 1 & 3S - 2 \\
S - 1 & 4S + 3
\end{bmatrix}
$$

$$
\det(A(S)) = (2S + 1)(4S + 3) - (3S - 2)(S - 1)
$$

Step 2: Expand and simplify:

$$
(2S + 1)(4S + 3) = 8S^2 + 6S + 4S + 3 = 8S^2 + 10S + 3
$$

$$
(3S - 2)(S - 1) = 3S \times S - 3S \times 1 - 2 \times S + 2 \times 1 = 3S^2 - 3S - 2S + 2 = 3S^2 - 5S + 2
$$

Step 3: Compute the determinant:

$$
\det(A(S)) = 8S^2 + 10S + 3 - (3S^2 - 5S + 2) = (8S^2 - 3S^2) + (10S + 5S) + (3 - 2) = 5S^2 + 15S + 1
$$

Step 4: Find S where the determinant equals zero:

$$
5S^2 + 15S + 1 = 0
$$

Divide throughout by 5:

$$
S^2 + 3S + \frac{1}{5} = 0
$$

Use quadratic formula:

$$
S = \frac{-3 \pm \sqrt{9 - 4 \times 1 \times \frac{1}{5}}}{2} = \frac{-3 \pm \sqrt{9 - \frac{4}{5}}}{2} = \frac{-3 \pm \sqrt{\frac{45}{5} - \frac{4}{5}}}{2} = \frac{-3 \pm \sqrt{\frac{41}{5}}}{2}
$$

Thus, the system has a unique solution for all S except at:

$$
S = \frac{-3 \pm \sqrt{\frac{41}{5}}}{2}
$$

where the determinant vanishes, and the system may be inconsistent or have infinitely many solutions.

Interpreting the Results and Additional Conditions

When the Determinant is Non-Zero

For all values of S that do not satisfy the roots of the determinant equation, the coefficient matrix is invertible. In these cases, the system has a unique solution, which can be explicitly computed using methods like Cramer's rule or matrix inversion.

When the Determinant is Zero

At values of S satisfying $\det(A(S))=0$, further analysis is required:

- Check the consistency conditions: substitute the particular S into the system and see if the equations are compatible.
- Determine whether the system has infinitely many solutions: if the equations reduce to the same equation, the solution set is infinite.
- Identify inconsistency: if the equations contradict, then no solutions exist at that S.

Practical Implications

Understanding the range of S where solutions exist is crucial in applications:

- Engineering: ensuring parameters stay within safe bounds to guarantee a unique solution for stress or load calculations.
- Economics: modeling scenarios where certain parameters lead to unique equilibria versus multiple or no equilibria.
- Computer Science: designing algorithms stable under parameter variations.

Extending to Larger and More Complex Systems

While the example focuses on a 2x2 system, the principles extend naturally to larger systems:

- The key is analyzing the determinant of the coefficient matrix.
- For larger matrices, computing determinants and solving polynomial equations in parameters can be more challenging but follows similar logic.
- Advanced methods, such as eigenvalue analysis or rank conditions, become invaluable.

Summary: The Path to Finding the Critical Values of S

To determine the values of S for which the system has a unique solution, follow these steps:

1. Identify the coefficient matrix $A(S)$ and the constant vector $\mathbf{b}(S)$.
2. Compute the determinant $\det(A(S))$ as a function of S.
3. Solve $\det(A(S)) \neq 0$ to find the set of S where the system is invertible.
4. Examine the roots of $\det(A(S)) = 0$ to understand where the solution nature changes.
5. Verify additional conditions for consistency at these critical S values, as zero

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