Determine Whether Or Not F Is A Conservative Vector Field. If It Is, Find A Function F Such That F =

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Understanding whether a vector field is conservative is a fundamental concept in vector calculus, with wide-ranging applications in physics, engineering, and mathematics. A conservative vector field has properties that simplify the analysis of the field, such as the existence of a potential function. This article provides a comprehensive guide to determining if a given vector field $\mathbf{F}$ is conservative and, if so, how to find an associated potential function ϕ such that $\mathbf{F} = \nabla \phi$.

What Is a Conservative Vector Field?

A vector field $\mathbf{F}$ in a domain $D \subseteq \mathbb{R}^n$ (commonly $\mathbb{R}^2$ or $\mathbb{R}^3$) is called conservative if there exists a scalar potential function $\phi: D \to \mathbb{R}$ such that:

$$
\mathbf{F} = \nabla \phi
$$

where $\nabla \phi$ is the gradient of ϕ. The key characteristics of a conservative vector field include:

- Path Independence: The line integral of $\mathbf{F}$ between two points is independent of the path taken.
- Zero Curl: In simply connected domains, the curl of $\mathbf{F}$ is zero:

$$
\nabla \times \mathbf{F} = \mathbf{0}
$$

- Existence of Potential Function: There exists ϕ such that $\mathbf{F} = \nabla \phi$.

Criteria for a Vector Field to Be Conservative

In order to determine if a vector field $\mathbf{F}$ is conservative, certain conditions must be satisfied depending on the domain:

1. Simply Connected Domain

- For $\mathbf{F}$ to be conservative, the domain D must be simply connected; that is, any closed loop in D can be continuously contracted to a point without leaving D.

2. Curl-Free Condition

- In $\mathbb{R}^3$, the necessary and sufficient condition (for simply connected domains) is:

$$
\nabla \times \mathbf{F} = \mathbf{0}
$$

- In $\mathbb{R}^2$, for a vector field $\mathbf{F} = (P, Q)$, the condition is:

$$
\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}
$$

3. Compatibility Conditions

- For higher dimensions, additional conditions involving partial derivatives ensure the field's conservative nature.

Step-by-Step Method to Determine if $\mathbf{F}$ Is Conservative

To verify whether a vector field $\mathbf{F}$ is conservative, follow these steps:

Step 1: Identify the Vector Field Components

- Express $\mathbf{F}$ explicitly, for example:

$$
\mathbf{F}(x, y, z) = P(x, y, z) \mathbf{i} + Q(x, y, z) \mathbf{j} + R(x, y, z) \mathbf{k}
$$

Step 2: Check the Domain

- Ensure the domain D where $\mathbf{F}$ is defined is simply connected. If not, the field may not be conservative even if curl conditions are satisfied.

Step 3: Compute the Curl of $\mathbf{F}$

- Calculate:

$$
\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
$$

$$

- If $\nabla \times \mathbf{F} = \mathbf{0}$ everywhere in D, then $\mathbf{F}$ is conservative (assuming D is simply connected).

Step 4: Verify the Conditions

- Confirm the curl is zero throughout the domain.
- If the curl is non-zero at any point, $\mathbf{F}$ is not conservative.

Step 5: Find the Potential Function ϕ

- If $\mathbf{F}$ is conservative, proceed to find ϕ, solving:

$$
\nabla \phi = \mathbf{F}
$$

- Integrate component-wise:

- Integrate P with respect to x:

$$
\phi(x, y, z) = \int P(x, y, z) \, dx + C(y, z)
$$

- Differentiate ϕ with respect to y and compare with Q:

$$
\frac{\partial \phi}{\partial y} = Q(x, y, z)
$$

- Similarly, integrate R with respect to z if needed.

- Solve for the functions $C(y, z)$, $C(z)$, etc., to determine the full potential function.

Practical Examples

To solidify understanding, consider the following examples:

Example 1: $\mathbf{F} = (2x, 2y)$

- Step 1: Components $P = 2x$, $Q = 2y$.
- Step 2: Domain $\mathbb{R}^2$ (simply connected).
- Step 3: Compute $\frac{\partial Q}{\partial x} = 0$, $\frac{\partial P}{\partial y} = 0$. Since they are equal, the curl condition is satisfied.
- Step 4: Confirmed $\mathbf{F}$ is conservative.
- Step 5: Find ϕ:

$$
\phi(x, y) = \int 2x \, dx = x^2 + C(y)
$$

$$
\frac{\partial \phi}{\partial y} = C'(y) = 2y \Rightarrow C'(y) = 2y
$$

$$
C(y) = y^2 + \text{constant}
$$

- Final potential function:

$$
\boxed{\phi(x, y) = x^2 + y^2 + \text{constant}}
$$

Example 2: $\mathbf{F} = (y, -x)$

- Step 1: $P = y$, $Q = -x$.
- Step 2: Domain $\mathbb{R}^2$, simply connected.
- Step 3: $\frac{\partial Q}{\partial x} = -1$, $\frac{\partial P}{\partial y} = 1$. Since these are not equal, $\nabla \times \mathbf{F} \neq 0$.
- Step 4: The field is not conservative.

Significance of Conservative Vector Fields in Physics and Engineering

Conservative vector fields are essential in physical contexts such as:

- Gravitational Fields: Potential energy functions exist for gravitational forces.
- Electrostatics: Electric fields are conservative; potential functions define voltage.
- Fluid Dynamics: Velocity fields can be conservative, simplifying flow analysis.

In engineering, identifying conservative fields allows for the calculation of potential energy, work done, and simplifies the analysis of systems.

Summary and Key Takeaways

- A vector field $\mathbf{F}$ is conservative if it can be expressed as the gradient of a scalar potential function ϕ.
- The key criteria involve checking the curl of $\mathbf{F}$ and the domain's topology.
- The process involves component identification, curl computation, and potential function derivation.
- Not all vector fields are conservative; non-zero curl indicates non-conservative nature.
- Conservative fields facilitate easier computation of line integrals, work, and energy in physical systems.

Conclusion

Determining whether a vector field \(\mathbf{F} \) is conservative is a critical skill in vector calculus, particularly when analyzing physical systems and solving differential equations. By following the outlined criteria and methodical approach—checking the domain, computing curl, and constructing the potential

Frequently Asked Questions

How can I determine if a vector field F is conservative?

To determine if F is conservative, verify if its curl is zero (in 3D) or if the line integral around any closed loop is zero (in 2D). If either condition holds, F is conservative.

What is the significance of a vector field being conservative?

A conservative vector field implies the existence of a potential function F such that $F = \nabla \phi$. This simplifies computations of line integrals and helps in understanding energy conservation in physics.

How do I find a potential function for a conservative vector field?

You integrate the components of the vector field with respect to their variables, ensuring consistency among the partial derivatives, to construct the potential function ϕ.

Can a vector field be non-conservative if it is defined over a simply connected domain?

Yes, a vector field over a simply connected domain can be non-conservative if its curl is not zero, indicating no potential function exists.

Is the condition curl F = 0 sufficient for F to be conservative?

In simply connected domains, yes. If curl F = 0 everywhere and the domain is simply connected, then F is conservative.

What is the process to find the potential function F if F is conservative?

Start by integrating one component of F with respect to its variable, then determine the remaining parts by matching derivatives to the other components, ensuring consistency throughout.

What are common mistakes to avoid when determining if F is conservative?

Avoid assuming curl F = 0 guarantees conservativeness without checking the domain's connectivity, and ensure proper integration constants are considered when constructing potential functions.

If F = (P, Q), how do I verify if F is conservative in 2D?

Check if $\partial Q/\partial x = \partial P/\partial y$; if they are equal and the domain is simply connected, F is conservative, and you can find a potential function by integrating P and Q accordingly.

Can you give an example of finding a potential function for a conservative vector field?

Yes. For F = (2xy, x^2), since curl F = 0, integrate P=2xy with respect to x yielding φ = x^2 y + C. Then, verify by differentiating φ with respect to y to match Q, ensuring consistency.

Additional Resources

Determine Whether Or Not F Is A Conservative Vector Field. If It Is, Find A Function F Such That F =

Understanding the nature of vector fields is fundamental in various branches of physics and engineering, especially when analyzing force fields, potential functions, and energy landscapes. One key classification within vector calculus is identifying whether a given vector field is conservative. Recognizing a conservative vector field enables us to simplify complex problems, determine potential energy functions, and analyze flow patterns more effectively. This article delves into the criteria that determine whether a vector field is conservative and guides you through the process of finding the potential function if it exists.

What Is a Vector Field?

Before exploring whether a vector field is conservative, it's essential to understand what a vector field is.

A vector field assigns a vector to each point in a region of space. For example, in physics, the gravitational force field around a planet or the magnetic field around a magnet are vector fields. Mathematically, a vector field in three-dimensional space can be represented as:

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$$

where:

- $P(x, y, z)$, $Q(x, y, z)$, and $R(x, y, z)$ are scalar functions representing the components of the vector field along the x, y, and z axes, respectively.

Understanding Conservative Vector Fields

A vector field $\mathbf{F}$ is called conservative if it can be expressed as the gradient of some scalar potential function ϕ. Formally,

$$\mathbf{F} = \nabla \phi$$

which implies:

$$\mathbf{F} = \frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}$$

Why are conservative fields important?

- They have zero curl, meaning the field has no rotational component.
- They are path-independent; the work done moving along a path depends only on the endpoints, not the path taken.
- They allow the definition of a scalar potential energy function, simplifying analysis in physics and engineering.

Characterizing Conservative Fields

To determine whether a vector field $\mathbf{F}$ is conservative, mathematicians rely on certain conditions, primarily involving the curl and the domain of the field.

Key Conditions:

1. Curl-Free Condition:
In simply connected regions (regions without holes), a vector field $\mathbf{F}$ is conservative if and only if:

$$\nabla \times \mathbf{F} = \mathbf{0}$$

The curl of $\mathbf{F}$ measures its tendency to rotate around a point. Zero curl indicates no local rotation, a hallmark of conservative fields.

2. Path Independence and Domain Connectedness:
The domain of $\mathbf{F}$ must be simply connected. If the domain has holes or is not simply connected, a curl-free field may not be conservative.

Step-by-Step Procedure to Determine if $\mathbf{F}$ Is Conservative

Step 1: Compute the Curl of $\mathbf{F}$

Given $\mathbf{F} = P \mathbf{i} + Q \mathbf{j} + R \mathbf{k}$, compute:

$$
\nabla \times \mathbf{F} =
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
P & Q & R
\end{vmatrix}
$$

which expands to:

$$
\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
$$

If all three components of the curl are zero everywhere in the domain, $\mathbf{F}$ is conservative (assuming the domain is simply connected).

Step 2: Confirm the Domain is Simply Connected

Ensure that the region where $\mathbf{F}$ is defined has no holes or obstacles that could prevent the existence of a potential function.

Step 3: Find the Potential Function ϕ

If the previous conditions are satisfied, proceed to find ϕ:

- Integrate the P component with respect to x:

$$
\phi(x, y, z) = \int P(x, y, z) \, dx + C(y, z)
$$

- Differentiate ϕ with respect to y and equate to Q:

$$
\frac{\partial \phi}{\partial y} = Q(x, y, z) \Rightarrow \text{determine } C(y, z)
$$

- Similarly, differentiate ϕ with respect to z and set equal to R:

$$
\frac{\partial \phi}{\partial z} = R(x, y, z) \Rightarrow \text{determine the remaining parts}
$$

- Combine these integrations to find the explicit form of ϕ.

Practical Example: Is $\mathbf{F} = (2xy, x^2 + z, y)$ Conservative?

Let's illustrate the process with a concrete example.

Step 1: Identify the components

$$
P = 2xy, \quad Q = x^2 + z, \quad R = y
$$

Step 2: Compute the curl

$$
\nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}
$$

Calculate each component:

- $\frac{\partial R}{\partial y} = 1$
- $\frac{\partial Q}{\partial z} = 1$
- $\frac{\partial P}{\partial z} = 0$
- $\frac{\partial R}{\partial x} = 0$
- $\frac{\partial Q}{\partial x} = 2x$
- $\frac{\partial P}{\partial y} = 2x$

Thus,

$$
\nabla \times \mathbf{F} = (1 - 1)\mathbf{i} + (0 - 0)\mathbf{j} + (2x - 2x)\mathbf{k} = \mathbf{0}
$$

Step 3: Confirm the domain

Assuming the domain is all of $\mathbb{R}^3$, which is simply connected.

Conclusion: Since the curl is zero everywhere and the domain is simply connected, $\mathbf{F}$ is conservative.

Step 4: Find the potential function ϕ

- Integrate $P = 2xy$ with respect to x:

$$
\phi(x, y, z) = \int 2xy \, dx = x^{2} y + C(y, z)
$$

- Differentiate ϕ with respect to y:

$$
\frac{\partial \phi}{\partial y} = x^{2} + \frac{\partial C(y, z)}{\partial y}
$$

Set equal to $Q = x^{2} + z$:

$$
x^{2} + \frac{\partial C(y, z)}{\partial y} = x^{2} + z \Rightarrow \frac{\partial C(y, z)}{\partial y} = z
$$

Integrate with respect to y:

$$
C(y, z) = y z + D(z)
$$

- Now, differentiate ϕ with respect to z:

$$
\frac{\partial \phi}{\partial z} = \frac{\partial C(y, z)}{\partial z} = y + \frac{d D(z)}{d z}
$$

Set equal to $R = y$:

$$
y + \frac{d D(z)}{d z} = y \Rightarrow \frac{d D(z)}{d z} = 0
$$

Thus, $D(z)$ is a constant, which we can ignore.

Final potential function:

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\phi(x, y,

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