Evaluate The Iterated Integral By Converting To Polar Coordinates. ./2 - Y2 5(x + Y) Dx Dy 12- 2v2 3

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When it comes to evaluating complex double integrals, especially those involving non-trivial regions or integrand functions, converting to polar coordinates often simplifies the process significantly. This approach is particularly useful when the region of integration is circular, elliptical, or can be more easily described in terms of radius and angle rather than Cartesian coordinates. Today, we will explore how to evaluate the iterated integral given by:

$$
\iint_{R} (\text{expression}) \, dx\,dy
$$

where the integral's bounds and the integrand suggest a transformation to polar coordinates, making the evaluation more straightforward. The specific integral in question appears to involve terms such as Y^2, $x + Y$, and constants like 12 and $2\sqrt{3}$. Although the original notation appears somewhat fragmented, we will interpret and reconstruct the integral carefully to guide you through the process.

Understanding the Integral and Its Region of Integration

Before converting to polar coordinates, it’s crucial to interpret the integral's bounds and integrand correctly.

1. Deciphering the Integral Expression

The notation appears to be:

$$
\iint_{D} \left(2 - y^2 + 5(x + y) \right) \, dx\,dy
$$

or similar, with bounds related to 12 and $2\sqrt{3}$. Since the given expression is somewhat ambiguous, we'll assume the integral involves the function:

$$
f(x, y) = 2 - y^2 + 5(x + y)
$$

over a region D. The bounds possibly relate to the region's limits, which might be a circle or a sector involving radii or angles corresponding to these constants.

2. Determining the Region D

Given the constants, a common region associated with such integrals involves circles or sectors:

- The constant 12 could relate to a radius squared, i.e., $r^2 = 12$, implying $r = \sqrt{12} = 2\sqrt{3}$.
- The mention of $2\sqrt{3}$ supports this, suggesting the region might be a circle with radius $2\sqrt{3}$.

Assuming the region D is a circle centered at the origin with radius $2\sqrt{3}$:

$$
D: \quad x^2 + y^2 \leq 12
$$

which is a common scenario for converting to polar coordinates.

Converting Cartesian Coordinates to Polar Coordinates

Conversion to polar coordinates is a standard technique when dealing with circular regions, as it simplifies the bounds and often the integrand.

1. Polar Coordinate Definitions

The transformation from Cartesian (x, y) to polar (r, θ) is given by:

$$

$$
x = r \cos \theta
$$

$$
y = r \sin \theta
$$

where:

- $r \geq 0$ is the distance from the origin.
- $\theta \in [0, 2\pi)$ is the angle measured from the positive x-axis.

2. Jacobian for the Transformation

The differential area element transforms as:

$$
dx\,dy = r\,dr\,d\theta
$$

This Jacobian determinant accounts for the change of variables from Cartesian to polar coordinates.

3. Expressing the Integrand in Polar Coordinates

Given the integrand:

$$
f(x, y) = 2 - y^2 + 5(x + y)
$$

substituting $x = r \cos \theta$ and $y = r \sin \theta$:

$$
f(r, \theta) = 2 - (r \sin \theta)^2 + 5(r \cos \theta + r \sin \theta)
$$

which simplifies to:

$$
f(r, \theta) = 2 - r^2 \sin^2 \theta + 5 r (\cos \theta + \sin \theta)
$$

Setting Up the Integral in Polar Coordinates

Having established the transformation, the integral becomes:

$$
\iint_{D} f(x, y) \, dx\,dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{R} f(r, \theta) \, r \, dr\, d\theta
$$

where $R = 2\sqrt{3}$.

1. Final Form of the Integral

$$
\boxed{
\int_{0}^{2\pi} \int_{0}^{2 \sqrt{3}} \left[2 - r^2 \sin^2 \theta + 5 r (\cos \theta + \sin \theta) \right] r \, dr\, d\theta
}
$$

This integral can now be tackled by integrating with respect to r first, then θ.

Evaluating the Integral Step-by-Step

Let's proceed systematically to evaluate the integral.

1. Expand the Integrand

The integrand:

$$
f(r, \theta) \times r = \left[2 - r^2 \sin^2 \theta + 5 r (\cos \theta + \sin \theta) \right] r
$$

becomes:

$$
I(r, \theta) = 2r - r^3 \sin^2 \theta + 5 r^2 (\cos \theta + \sin \theta)
$$

2. Integrate with Respect to r

The limits are from 0 to $2\sqrt{3}$:

$$
\int_{0}^{2\sqrt{3}} \left(2r - r^3 \sin^2 \theta + 5 r^2 (\cos \theta + \sin \theta) \right) dr
$$

Break this into three separate integrals:

$$
I_1(\theta) = \int_{0}^{2\sqrt{3}} 2r\, dr
$$
$$
I_2(\theta) = - \sin^2 \theta \int_{0}^{2\sqrt{3}} r^3\, dr
$$
$$
I_3(\theta) = 5 (\cos \theta + \sin \theta) \int_{0}^{2\sqrt{3}} r^2\, dr
$$

Calculating each:

- $I_1(\theta)$:

$$
\int_{0}^{a} 2r\, dr = r^2 \bigg|_{0}^{a} = a^2
$$
with $a = 2 \sqrt{3}$:

$$
a^2 = (2 \sqrt{3})^2 = 4 \times 3 = 12
$$
So,

$$
I_1(\theta) = 12
$$

- $I_2(\theta)$:

$$
- \sin^2 \theta \times \frac{r^4}{4} \bigg|_{0}^{2 \sqrt{3}} = - \sin^2 \theta \times \frac{(2 \sqrt{3})^4}{4}
$$

Calculate $(2 \sqrt{3})^4$:

$$
(2 \sqrt{3})^4 = ((2)^4) \times (\sqrt{3})^4 = 16 \times (3^2) = 16 \times
$$

$$
9 = 144
$$

Thus,

$$
I_2(\theta) = - \sin^2 \theta \times \frac{144}{4} = - \sin^2 \theta \times 36
$$

- $I_3(\theta)$:

$$
5 (\cos \theta + \sin \theta) \times \frac{r^3}{3} \bigg|_{0}^{2 \sqrt{3}} = 5 (\cos \theta + \sin \theta) \times \frac{(2 \sqrt{3})^3}{3}
$$

Calculate $(2 \sqrt{3})^3$:

$$
(2 \sqrt{3})^3 = 2^3 \times (\sqrt{3})^3 = 8 \times 3 \sqrt{3} = 8 \times 3 \sqrt{3} = 24 \sqrt{3}
$$

Therefore:

$$
I_3(\theta) = 5 (\cos \theta + \sin \theta) \times \frac{24 \sqrt{3}}{3} = 5 (\cos \theta + \sin \theta) \times 8 \sqrt{3}
$$

Simplify:

$$
I_3(\theta) = 40 \sqrt{3} (\cos \theta + \sin \theta)
$$

---

3. Summing the Results

The total integral over r for a fixed θ:

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Frequently Asked Questions

What is the first step in converting the given iterated integral to polar coordinates?

The first step is to express the limits and the integrand in terms of polar coordinates, where $x = r \cos \theta$ and $y = r \sin \theta$, and then rewrite the bounds accordingly.

How do you determine the new limits of integration when converting from Cartesian to polar coordinates?

You analyze the original region described by the Cartesian bounds and express those bounds in terms of r and θ, often converting inequalities involving x and y into inequalities involving r and θ.

What is the significance of the Jacobian in converting the integral to polar coordinates?

The Jacobian accounts for the change of variables from (x, y) to (r, θ), and in this case, it is r, which must be multiplied with the integrand when changing variables.

How do you convert the integrand (2 - y^2) in the integral to polar coordinates?

Replace y with $r \sin \theta$, so y^2 becomes $(r \sin \theta)^2 = r^2 \sin^2 \theta$, and substitute into the integrand to get $2 - r^2 \sin^2 \theta$.

What are the main challenges in evaluating the given integral after converting to polar coordinates?

The main challenges include accurately determining the bounds in r and θ, simplifying the integrand involving r and θ, and integrating complex expressions like $r^3 \sin^2 \theta$.

How can symmetry help in evaluating the integral after converting to polar coordinates?

Symmetry can simplify the integral if the region or the integrand is symmetric, allowing certain parts of the integral to be evaluated as zero or combined to reduce computation.

What is the purpose of converting the integral to polar coordinates in this context?

Converting to polar coordinates simplifies the evaluation of integrals over regions with circular or radial symmetry, making the integrals more manageable.

Can you briefly outline the steps to evaluate the integral after converting to polar coordinates?

Yes, first convert the region to polar bounds, rewrite the integrand in terms of r and θ, include the Jacobian r, set up the new integral with these bounds, and then perform the integration with respect to r and θ.

Additional Resources

Evaluating the Iterated Integral by Converting to Polar Coordinates: A Comprehensive Guide

When faced with complex double integrals, particularly those involving non-Cartesian regions or integrand functions that are difficult to integrate directly, mathematicians and students alike turn to coordinate transformations. Among these, polar coordinates stand out as an elegant and powerful technique, especially for regions with circular symmetry or boundaries described by arcs and circles. This article offers an in-depth exploration of how to evaluate an iterated integral by converting it to polar coordinates, using a specific integral as our case study.

Understanding the Context: The Integral in Question

Before diving into the method, let's clarify the integral we are examining:

$$
\iint_{R} (2 - y^2) \, dx \, dy
$$

where the region R is described by the inequalities:

$$
12 - 2\sqrt{2} \leq x + y \leq 5
$$

and perhaps additional bounds implied by the problem statement, involving the variables x and y.

Note: The integral appears to involve a quadratic term y^2 and linear combinations $x + y$, which suggests that a coordinate transformation might simplify the bounds or the integrand.

Why Convert to Polar Coordinates?

Polar coordinates (r, θ) are defined as:

$$
x = r \cos \theta, \quad y = r \sin \theta
$$

The motivation for converting integrals into polar coordinates includes:

- Simplifying regions bounded by circles, arcs, or sectors.
- Transforming complicated bounds involving x and y into more manageable bounds in r and θ.
- Simplifying integrand functions involving $x^2 + y^2$ or radial symmetry.

In our case, the presence of y^2 and the linear combination $x + y$ suggests that the boundary regions involve circles or lines that become more tractable in polar coordinates.

Step-by-Step Approach to Conversion and Evaluation

1. Identifying the Region R

The first critical step is to understand the region over which the integral is taken. Based on the inequalities:

$$
12 - 2\sqrt{2} \leq x + y \leq 5
$$

and possibly other bounds, the region may be bounded by lines and circles.

- The inequalities involving $x + y$ define a strip between two parallel lines.
- The quadratic y^2 suggests potential circular boundaries if combined with other constraints, or perhaps the integrand's form hints at symmetry.

Key observations:

- The region could be a sector or a region bounded by lines and circles.
- The linear bounds $x + y = c$ become simple in polar coordinates:

$$
x + y = r \cos \theta + r \sin \theta = r (\cos \theta + \sin \theta)
$$

which simplifies the boundary conditions.

2. Expressing the Boundaries in Polar Coordinates

Transform the bounds:

- For $x + y$:

$$
x + y = r (\cos \theta + \sin \theta)
$$

- The inequalities:

$$
12 - 2\sqrt{2} \leq r (\cos \theta + \sin \theta) \leq 5
$$

- The quadratic term in y^2:

$$
y^2 = r^2 \sin^2 \theta
$$

which appears directly in the integrand.

3. Transforming the Integrand

The integrand:

$$
2 - y^2 = 2 - r^2 \sin^2 \theta
$$

becomes a function of r and θ.

The differential area element:

$$
dx \, dy = r \, dr \, d\theta
$$

due to the Jacobian of the polar coordinate transformation.

4. Setting Up the Integral in Polar Coordinates

Putting all parts together, the double integral becomes:

$$
\iint_{R} (2 - y^2) \, dx \, dy = \int_{\theta_{min}}^{\theta_{max}} \int_{r_{min}(\theta)}^{r_{max}(\theta)} \left(2 - r^2 \sin^2 \theta \right) r \, dr \, d\theta
$$

where:

- $r_{min}(\theta)$ and $r_{max}(\theta)$ are derived from the bounds on $x + y$:

$$
r (\cos \theta + \sin \theta) \in [12 - 2\sqrt{2}, 5]
$$

thus:

$$
r_{min}(\theta) = \frac{12 - 2\sqrt{2}}{\cos \theta + \sin \theta}
$$
$$
r_{max}(\theta) = \frac{5}{\cos \theta + \sin \theta}
$$

- The limits on θ depend on where the inequalities hold true and whether the denominators are positive or negative, which influences the bounds.

5. Determining the Angular Limits

The bounds on θ are determined by the region's geometry. For example:

- When $\cos \theta + \sin \theta > 0$, the bounds on r are positive.
- The values of θ satisfying the inequalities can be found by solving:

$$
r_{min}(\theta) \geq 0 \quad \text{and} \quad r_{max}(\theta) \geq r_{min}(\theta)
$$

This process may involve solving inequalities involving θ, considering the periodicity of sine and cosine.

Executing the Integral: Step-by-Step Calculation

1. Substituting and Simplifying

The integral becomes:

$$
\int_{\theta_{a}}^{\theta_{b}} \int_{r_{min}(\theta)}^{r_{max}(\theta)} \left(2r - r^3 \sin^2 \theta \right) dr \, d\theta
$$

which can be split into two integrals:

$$
\int_{\theta_{a}}^{\theta_{b}} \left[\int_{r_{min}(\theta)}^{r_{max}(\theta)} 2r \, dr - \int_{r_{min}(\theta)}^{r_{max}(\theta)} r^3 \sin^2 \theta \, dr \right] d\theta
$$

2. Computing the Inner Integrals

- First integral:

$$
\int 2r \, dr = r^2
$$

- Second integral:

$$
\int r^3 \, dr = \frac{r^4}{4}
$$

including the $\sin^2 \theta$ term as a constant with respect to r:

$$
\sin^2 \theta \int r^3 \, dr = \frac{\sin^2 \theta}{4} r^4
$$

Thus, the inner integral evaluates to:

$$
r_{max}^2(\theta) - r_{min}^2(\theta) - \frac{\sin^2 \theta}{4} \left(r_{max}^4(\theta) - r_{min}^4(\theta) \right)
$$

3. Integrating Over θ

Finally, the integral reduces to:

$$
\int_{\theta_{a}}^{\theta_{b}} \left[r_{max}^2(\theta) - r_{min}^2(\theta) - \frac{\sin^2 \theta}{4} \left(r_{max}^4(\theta) - r_{min}^4(\theta) \right) \right] d\theta
$$

which can be evaluated numerically or analytically, depending on the explicit bounds.

Practical Tips for Evaluating Such Integrals

- Understand the region thoroughly: Sketch the region bounded by the inequalities to avoid errors in bounds.
- Express boundaries explicitly: Write inequalities in terms of r and θ.
- Check for symmetry: Symmetries in the region or integrand can simplify calculations.
- Divide the region if necessary: For complicated regions, split into subregions with simpler bounds.
- Use substitution and identities: Trigonometric identities can simplify the integrand involving $\sin^2 \theta$ or $\cos \theta$.

Conclusion: The Power of Coordinate Transformation

Transforming an iterated integral into polar coordinates can drastically simplify the evaluation process, especially for regions involving circular boundaries or linear inequalities that become straightforward in the \(r, \theta \) plane. In our case study, rewriting the bounds and the integrand in terms of \(r \) and

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