Evaluate The Line Integral, Where C Is The Given Curve. C Xy Ds, C Is The Right Half Of The Circle X

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Introduction to Line Integrals

In multivariable calculus, line integrals are a powerful tool used to evaluate various physical and mathematical quantities along a curve in space. They extend the concept of definite integrals from functions of a single variable to functions defined along a path or curve in the plane or space. Line integrals have applications in physics, engineering, and other sciences, especially in fields such as electromagnetism, fluid dynamics, and work calculations.

This article focuses on evaluating the line integral of the form:

$$
\int_C xy\, ds
$$

where C is a specific curve, namely the right half of the circle x, and ds is the differential arc length along C. We aim to understand how to approach this problem systematically, including parameterization of the curve, the computation of ds, and the evaluation of the integral.

Understanding the Curve C: The Right Half of the Circle x

Definition of the Curve C

The problem states that C is the right half of the circle x, which suggests that the curve is a semicircular arc lying on the right side of a circle in the xy-plane.

To clarify, the typical circle is represented by the equation:

$$
x^2 + y^2 = r^2
$$

where r is the radius. The right half of this circle corresponds to the set of points where:

$$
x \geq 0 \quad \text{and} \quad x^2 + y^2 = r^2
$$

This semicircular arc extends from the top point $(0, r)$ to the bottom point $(0, -r)$, passing through the rightmost point $(r, 0)$.

Choosing the Radius and Parameterization

For simplicity, assume the circle has a radius $r = 1$. This choice simplifies calculations without loss of generality because the integral can be scaled accordingly.

The right half of the circle $x^2 + y^2 = 1$ can be parameterized as:

$$
x(t) = \cos t, \quad y(t) = \sin t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
$$

This parameterization traverses the right semicircle starting from the bottom point $(0, -1)$ at $t = -\frac{\pi}{2}$, moving through the rightmost point $(1, 0)$ at $t = 0$, and ending at the top point $(0, 1)$ at $t = \frac{\pi}{2}$.

Parameterization of the Curve C

Standard Parameterization

Given the circle of radius 1, the parametric equations for the right half are:

$$
x(t) = \cos t, \quad y(t) = \sin t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
$$

This parameterization covers the entire right semicircle, starting from the bottom to the top.

Computing the Differential Arc Length ds

The differential arc length ds along the curve is given by:

$$
ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt
$$

Calculating derivatives:

$$
\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t
$$

Substituting into the expression for ds:

$$
ds = \sqrt{(-\sin t)^2 + (\cos t)^2} dt = \sqrt{\sin^2 t + \cos^2 t} dt = 1 \cdot dt
$$

Thus, for the unit circle, the arc length differential simplifies to:

$$
ds = dt
$$

Formulating the Line Integral

The original line integral is:

$$
\int_C xy\, ds
$$

Using the parameterization:

$$
x(t) = \cos t, \quad y(t) = \sin t, \quad ds = dt
$$

we rewrite the integral as:

$$
\int_{t = -\pi/2}^{\pi/2} x(t) y(t) dt = \int_{-\pi/2}^{\pi/2} (\cos t)(\sin t) dt
$$

which simplifies to:

$$
\int_{-\pi/2}^{\pi/2} \cos t \sin t\, dt
$$

Evaluating the Integral

Using Trigonometric Identities

Recall the double-angle identity:

$$
\sin 2t = 2 \sin t \cos t
$$

Therefore:

$$
\cos t \sin t = \frac{1}{2} \sin 2t
$$

The integral becomes:

$$
\int_{-\pi/2}^{\pi/2} \frac{1}{2} \sin 2t\, dt
$$

Factoring out the constant:

$$
\frac{1}{2} \int_{-\pi/2}^{\pi/2} \sin 2t\, dt
$$

Computing the Integral

The integral of $\sin 2t$ with respect to t:

$$
\int \sin 2t\, dt = -\frac{1}{2} \cos 2t + C
$$

Applying limits:

$$
\frac{1}{2} \left[-\frac{1}{2} \cos 2t \right]_{t = -\pi/2}^{\pi/2} = -\frac{1}{4} \left[\cos 2t \right]_{-\pi/2}^{\pi/2}
$$

Calculate the cosine at the bounds:

- At $t = \pi/2$:

$$
\cos 2 \times \pi/2 = \cos \pi = -1
$$

- At $t = -\pi/2$:

$$
\cos 2 \times (-\pi/2) = \cos (-\pi) = \cos \pi = -1
$$

Subtracting:

$$
\left[\cos 2t \right]_{-\pi/2}^{\pi/2} = (-1) - (-1) = 0
$$

Hence, the value of the integral:

$$
-\frac{1}{4} \times 0 = 0
$$

Conclusion: Value of the Line Integral

The evaluated line integral:

$$
\int_C xy\, ds = 0
$$

This result indicates that, along the right half of the circle of radius 1, the integral of $xy \, ds$ is zero. The symmetry of the integrand and the curve leads to cancellation over the interval.

Generalization to Other Circles and Curves

While this example uses a circle of radius 1, the method extends to circles of any radius r. The key steps involve:

1. Parameterizing the curve: For a circle of radius r, the parameterization becomes:

$$
x(t) = r \cos t, \quad y(t) = r \sin t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]
$$

2. Computing ds: For radius r,

$$
\frac{dx}{dt} = -r \sin t, \quad \frac{dy}{dt} = r \cos t
$$

$$
ds = \sqrt{(-r \sin t)^2 + (r \cos t)^2} dt = r \sqrt{\sin^2 t + \cos^2 t} dt = r\, dt
$$

3. Formulating the integral:

$$
\int_{-\pi/2}^{\pi/2} (r \cos t)(r \sin t) \times r\, dt = r^3 \int_{-\pi/2}^{\pi/2} \cos t \sin t\, dt
$$

which simplifies similarly, leading to the same conclusion that the integral equals zero due to symmetry.

Additional Insights and Applications

Physical Interpretation

Line integrals like $\int_C xy\, ds$ are often interpreted as the work done by a force field along a path or the flux of a vector field across a curve. The symmetry leading to zero in this case indicates that the positive contributions of xy on one side of the curve are canceled out by the negative contributions on the opposite side.

Applications in Physics and Engineering

- Work Done

Frequently Asked Questions

What is the general approach to evaluate the line integral of a vector field over a curve C?

The general approach involves parameterizing the curve C, computing the differential element ds or dr, and then integrating the dot product of the vector field with the differential element over the parameter range.

How do you parameterize the right half of the circle x in the given curve C?

For the right half of the circle x, the circle equation is $x^2 + y^2 = r^2$. Parameterizing the right half, we can set $x = r \cos t$, $y = r \sin t$ with t from $-\pi/2$ to $\pi/2$, or use $x = \cos t$, $y = \sin t$ with t from $-\pi/2$ to $\pi/2$ if the circle has radius 1.

What is the significance of the direction of the curve C when evaluating the line integral?

The direction determines the sign and orientation of the parameterization, which affects the limits of integration and the sign of the computed integral. Consistent orientation ensures correct evaluation.

How is the differential element ds computed for the curve C?

The differential element ds can be computed as $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\, dt$ when parameterizing the curve, representing the infinitesimal arc length.

What are common mistakes to avoid when evaluating line integrals over curved paths like the right half of a circle?

Common mistakes include incorrect parameterization, forgetting to account for the direction of the curve, neglecting the differential element, or miscomputing the limits of integration.

How can Green’s theorem be used to evaluate line integrals over the curve C?

Green’s theorem relates the line integral around a closed curve to a double integral over the region it encloses. If C bounds a region, the line integral of a vector field can sometimes be converted into a double integral for easier evaluation.

In the context of the given curve C as the right half of the circle, how does symmetry simplify the evaluation of the integral?

Symmetry can simplify the integral by allowing certain terms to cancel out or by reducing the integral to a known form, especially when the integrand is symmetric or antisymmetric with respect to the axis of symmetry.

What are practical applications of evaluating line integrals over curved paths like the right half of a circle?

Practical applications include calculating work done by a force field along a path, flux calculations, electromagnetism problems, and fluid flow analysis where the path of integration is curved.

Additional Resources

Line Integral Evaluation: An In-Depth Analysis of the Curve C — The Right Half of the Circle x

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Introduction to Line Integrals and Their Significance

Line integrals are fundamental tools in vector calculus, often used to compute the work done by a force field along a path, the circulation of a field around a curve, and various applications across physics and engineering. Unlike simple integrals over intervals, line integrals evaluate functions along curves in space, adding a layer of geometric complexity that makes them both fascinating and essential.

In this exploration, we focus on evaluating the line integral:

$$
\int_C \mathbf{X} \, y \, ds,
$$

where C is specifically the right half of the circle x, and ds denotes the differential arc length element along the curve. Our goal is to understand the detailed process of computing this integral, deciphering the nuances of parametrization, the role of the vector field, and the geometric properties of the curve.

Understanding the Curve C: The Right Half of the Circle x

Geometric Description of the Curve

The phrase "the right half of the circle x" suggests that our curve C is a semicircular arc lying on the right side of a circle. To be precise, suppose the circle is centered at the origin with radius r, described by:

$$
x^2 + y^2 = r^2.
$$

The "right half" of this circle includes all points where $x \geq 0$ on the circumference, meaning the semicircular arc from the top point $(0, r)$ to the bottom point $(0, -r)$, passing through points with positive x-coordinates.

Key properties:

- Center: $(0, 0)$
- Radius: r
- Domain: $x \geq 0$ on the circle
- Path C: The arc from $(0, r)$ to $(0, -r)$

Parametrization of the Curve C

To evaluate the line integral, the first step is to parametrize the curve C. Since C is the right half of the circle, a natural parametrization uses the standard parametric equations of a circle:

$$
x = r \cos t, \quad y = r \sin t,
$$

with the parameter t typically in the interval $[0, 2\pi]$.

Parametrization for the right half:

- The right half corresponds to $x \geq 0$, which implies $\cos t \geq 0$.
- On the circle, $\cos t \geq 0$ when $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Thus, the parametrization becomes:

$$
\boxed{
\begin{aligned}
x(t) &= r \cos t, \\
y(t) &= r \sin t,
\end{aligned}
}
\quad \text{for } t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].
$$

Differential element:

$$
ds = |\mathbf{r}'(t)| dt,
$$

where

$$
\mathbf{r}(t) = (x(t), y(t)),
$$

and

$$
\mathbf{r}'(t) = \left(-r \sin t, r \cos t\right).
$$

The magnitude:

$$
|\mathbf{r}'(t)| = \sqrt{(-r \sin t)^2 + (r \cos t)^2} = r \sqrt{\sin^2 t + \cos^2 t} = r.
$$

So,

$$
ds = r\, dt.
$$

Formulating the Line Integral

Given the vector field $\mathbf{X}$ and the integrand y, the line integral:

$$
\int_C \mathbf{X} \, y \, ds,
$$

requires clarification. Typically, a line integral over a curve C involving a vector field $\mathbf{X}$ and a scalar function y is expressed as:

$$
\int_C \mathbf{X} \cdot d\mathbf{r} \quad \text{or} \quad \int_C f(x, y) ds,
$$

depending on the context.

Assumption for clarity:

- Suppose the integral is of the form:

$$
\int_C \mathbf{X} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy,
$$

where $\mathbf{X} = (P, Q)$.

- Alternatively, if the integral is a scalar line integral:

$$
\int_C y\, ds,
$$

which represents integrating the scalar function y along the curve with respect to arc length.

Given the notation "C X y ds," it strongly suggests the scalar line integral:

$$
\int_C y\, ds,
$$

where y is the scalar function evaluated along the curve, integrated over the arc length.

Thus, the focus is on:

\[
\boxed{
I = \int_C y\, ds.

}
\]

Explicit Computation of the Line Integral \(I = \int_C y\, ds \)

Step 1: Substituting the parametrization

Using \(y(t) = r \sin t \) and \(ds = r\, dt \), the integral becomes:

\[
I = \int_{t = -\pi/2}^{\pi/2} y(t) |\mathbf{r}'(t)| dt = \int_{-\pi/2}^{\pi/2} r \sin t \times r\, dt = r^2 \int_{-\pi/2}^{\pi/2} \sin t\, dt.
\]

Step 2: Computing the integral over \(t \)

\[
I = r^2 \left[-\cos t \right]_{-\pi/2}^{\pi/2} = r^2 \left(-\cos \frac{\pi}{2} + \cos \left(-\frac{\pi}{2}\right) \right).
\]

Since \(\cos \frac{\pi}{2} = 0 \) and \(\cos(-\frac{\pi}{2}) = 0 \), we have:

\[
I = r^2 (0 + 0) = 0.
\]

Result:

\[
\boxed{
\int_C y\, ds = 0.
}
\]

This indicates that the integral of \(y \) over the right semicircular arc, with respect to arc length, sums to zero. Geometrically, this can be interpreted as the positive contributions of \(y \) in the upper half being canceled by the negative contributions in the lower half, considering the symmetry.

Alternative Approach: Incorporating a Vector Field $\mathbf{X}$

If the original integral involves a vector field $\mathbf{X} = (P, Q)$, then the line integral takes the form:

$$
I = \int_C \mathbf{X} \cdot d\mathbf{r} = \int_C P\, dx + Q\, dy.
$$

Suppose $\mathbf{X}$ is specified as:

$$
\mathbf{X} = (P(x,y), Q(x,y)),
$$

and the goal is to evaluate:

$$
\int_C P\, dx + Q\, dy,
$$

along the same curve C.

In such a case, the steps involve:

- Parametrizing $x(t)$ and $y(t)$.
- Computing $dx = x'(t) dt$ and $dy = y'(t) dt$.
- Substituting into the integral:

$$
I = \int_{t=-\pi/2}^{\pi/2} \left[P(x(t), y(t)) x'(t) + Q(x(t), y(t)) y'(t) \right] dt.
$$

- Evaluating the integral over t.

Without specific functions P and Q, the general approach remains the same: parametrize, substitute, and integrate.

Discussion on the Significance and Applications

Evaluating line integrals over specific curves like the right half of a circle is more than an academic exercise; it has practical implications across multiple disciplines:

- Physics: Calculating work done by force fields along specified paths.

- Engineering: Analyzing fluid flow along curved boundaries.
- Mathematics: Understanding properties of vector fields, such as conservative fields, curl, and divergence.

In our case, the symmetric nature of the circle simplifies the integral, often leading to results like zero, which carry meaningful interpretations regarding symmetry and field behavior.

Conclusion: Mastery of Curve Integration Techniques

Through this detailed examination

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