Evaluate The Surface Integral. (x + Y + 2) D5, S Is The Parallelogram With Parametric Equations Xu +

Evaluate The Surface Integral. (x + Y + 2) D5, S Is The Parallelogram With Parametric Equations Xu +

Introduction

Surface integrals are a fundamental concept in vector calculus, essential for calculating flux, mass, and other physical quantities over surfaces. When dealing with a scalar field, the surface integral allows us to evaluate the accumulation of a quantity across a surface in three-dimensional space. This article provides a comprehensive guide to evaluating the surface integral of the form:

$$
\iint_{S} (x + y + 2) \, dS
$$

where S is a parallelogram in space, defined via parametric equations, specifically involving the parameters u and v. We will delve into the steps to compute such integrals, interpret the parametric equations of the surface, and discuss how to set up and evaluate the integral systematically.

Understanding the Surface and Its Parametric Representation

The Parallelogram Surface S

A parallelogram in $\mathbb{R}^3$ can be described parametrically by two vectors originating from a point $\mathbf{r}_0$, say $\mathbf{a}$ and $\mathbf{b}$. The parametric equations for the surface are:

$$
\mathbf{r}(u,v) = \mathbf{r}_0 + u \mathbf{a} + v \mathbf{b}
$$

where u, v vary over a specific domain, often a rectangle $0 \leq u, v \leq 1$ or some other bounds depending on the problem.

Setting Up the Surface Integral

General Form

The surface integral of a scalar function $f(x, y, z)$ over a surface S is expressed as:

$$
\iint_{S} f(x, y, z) \, dS

\]

To evaluate this integral when S is parametrized by $\mathbf{r}(u, v)$, the formula becomes:

$$
\iint_{D} f(\mathbf{r}(u, v)) \, |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv
$$

where:

- D is the domain in the uv-plane (the parameter domain),
- $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$,
- $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$,
- $|\mathbf{r}_u \times \mathbf{r}_v|$ is the magnitude of the cross product of the partial derivatives, representing the differential surface area element dS.

Step-by-Step Evaluation Process

1. Identify the Parametric Equations

Suppose the surface S is given by:

$$
\mathbf{r}(u, v) = \mathbf{r}_0 + u \mathbf{a} + v \mathbf{b}
$$

with known vectors $\mathbf{a}$ and $\mathbf{b}$, and the parameter domain D.

For example, if:

$$
\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))
$$

then the coordinate functions are explicitly defined in terms of u and v.

2. Compute Partial Derivatives

Calculate:

$$
\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}
\quad \text{and} \quad
\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}
$$

These are vectors in $\mathbb{R}^3$.

3. Find the Cross Product

Compute:

$$
\mathbf{r}_u \times \mathbf{r}_v
$$

and determine its magnitude:

$$
|\mathbf{r}_u \times \mathbf{r}_v|
$$

which serves as the Jacobian determinant for the surface element.

4. Substitute into the Integral

Express the function $f(x, y, z) = x + y + 2$ in terms of u and v via the parametric equations:

$$
f(\mathbf{r}(u, v)) = x(u, v) + y(u, v) + 2
$$

Set up the double integral over the domain D:

$$
\iint_{D} \left(x(u, v) + y(u, v) + 2 \right) |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv
$$

Example: Calculating the Surface Integral Over a Parallelogram

Suppose the surface S is defined by the following parametric equations:

$$
\mathbf{r}(u, v) = \mathbf{r}_0 + u \mathbf{a} + v \mathbf{b}
$$

where:

- $\mathbf{r}_0 = (x_0, y_0, z_0)$,
- $\mathbf{a} = (a_x, a_y, a_z)$,
- $\mathbf{b} = (b_x, b_y, b_z)$,

and the parameters u, v vary over $0 \leq u, v \leq 1$.

Step-by-Step Calculation

Step 1: Determine $\mathbf{r}_u$ and $\mathbf{r}_v$

$$
\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \mathbf{a}
$$
$$
\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \mathbf{b}
$$

Step 2: Compute the Cross Product

$$
\mathbf{r}_u \times \mathbf{r}_v = \mathbf{a} \times \mathbf{b}
$$

The magnitude:

$$
|\mathbf{a} \times \mathbf{b}| = \sqrt{ (a_y b_z - a_z b_y)^2 + (a_z b_x - a_x b_z)^2 + (a_x b_y - a_y b_x)^2 }
$$

which is a scalar quantity representing the area scaling factor of the surface element.

Step 3: Express $f(x, y, z)$ in Terms of u and v

Using the parametric equations:

$$
x(u, v) = x_0 + u a_x + v b_x
$$
$$
y(u, v) = y_0 + u a_y + v b_y
$$
$$
z(u, v) = z_0 + u a_z + v b_z
$$

then:

$$
f(x, y, z) = x + y + 2 = (x_0 + u a_x + v b_x) + (y_0 + u a_y + v b_y) + 2
$$

Simplify:

$$
f(u, v) = (x_0 + y_0 + 2) + u (a_x + a_y) + v (b_x + b_y)
$$

Evaluation of the Integral

The surface integral becomes:

$$
\iint_{0}^{1} \int_{0}^{1} f(u, v) |\mathbf{a} \times \mathbf{b}| \, du \, dv
$$

which simplifies to:

$$
|\mathbf{a} \times \mathbf{b}| \int_{0}^{1} \int_{0}^{1} \left[(x_0 + y_0 + 2) + u (a_x + a_y) + v (b_x + b_y) \right] du \, dv
$$

This integral can be separated into parts:

$$
|\mathbf{a} \times \mathbf{b}| \left[(x_0 + y_0 + 2) \int_{0}^{1} \int_{0}^{1} du dv + (a_x + a_y) \int_{0}^{1} u du \int_{0}^{1} dv + (b_x + b_y) \int_{0}^{1} v dv \int_{0}^{1} du \right]
$$

Calculating each:

- $\int_{0}^{1} du = 1$
- $\int_{0}^{1} dv = 1$
- $\int_{0}^{1} u du = \frac{1}{2}$
- $\int_{0}^{1} v dv = \frac{1}{2}$

Putting it all together:

$$
|\mathbf{a} \times \mathbf{b}| \left[(x_0 + y_0 + 2) \times 1 + (a_x + a_y) \times \frac{1}{2} \times 1 + (b_x + b_y) \times \frac{1}{2} \times 1 \right]
$$

which

Frequently Asked Questions

What is the general approach to evaluate the surface integral of the function (x + y + 2) over a parallelogram surface defined parametrically?

To evaluate the surface integral over a parallelogram surface, first parametrize the surface using given parametric equations, then compute the surface element dS by finding the cross product of the partial derivatives of the parametrization. Next, set up the integral of the function multiplied by the magnitude of this cross product over the parameter domain.

How do you parametrize a parallelogram surface defined by the vector equations in terms of parameters u and v?

A parallelogram surface can be parametrized as $r(u,v) = r0 + ua + vb$, where r0 is a point on the surface and vectors a and b define the sides. The parameters u and v typically vary over a rectangle, such as $0 \leq u \leq 1$ and $0 \leq v \leq 1$, or as specified by the problem.

What is the significance of the cross product of partial derivatives in surface integrals?

The cross product of the partial derivatives $\partial r/\partial u$ and $\partial r/\partial v$ gives a vector normal to the surface element, and its magnitude represents the area element dS. This is essential for converting a surface integral into a double integral over the parameter domain.

How do you evaluate the integral of (x + y + 2) over a parallelogram using parametric equations?

Express x and y in terms of parameters u and v from the parametrization, write the integrand as a function of u and v, compute the cross product to find the differential surface element, then integrate the resulting expression over the parameter domain, typically a rectangle.

What are common mistakes to avoid when calculating surface integrals over parametrized surfaces?

Common mistakes include incorrect parametrization, forgetting to compute the magnitude of the cross product for the surface element, misidentifying the limits of integration, and neglecting the vector nature of the differential area element.

How does the given function (x + y + 2) relate to the parameters u and v in the surface integral?

Once the surface is parametrized as $x = x(u,v)$ and $y = y(u,v)$, substitute these expressions into the function to write it entirely in terms of u and v. This allows integrating over the parameter domain to evaluate the surface integral.

If the surface is a parallelogram defined by vectors a and b, how are the bounds of u and v determined?

The bounds depend on how the parametrization is set up. Typically, if u and v are scaled from 0 to 1, then the surface is described over $0 \leq u \leq 1$ and $0 \leq v \leq 1$. Adjustments are made based on the problem's specifics.

What is the role of the scalar function (x + y + 2) in the surface integral, and how does it affect the computation?

The scalar function serves as the integrand, representing the quantity being summed over the surface. Its values are evaluated at each point on the surface, influencing the integral's result by weighting different surface regions differently.

Can you provide a step-by-step outline for evaluating the surface integral of (x + y + 2) over the given parallelogram?

Yes. Step 1: Parametrize the surface as $r(u,v)$. Step 2: Find $\partial r/\partial u$ and $\partial r/\partial v$. Step 3: Compute the cross product to find the surface element magnitude. Step 4: Express x and y in terms of u and v. Step 5: Set up the double integral of $(x(u,v) + y(u,v) + 2)$ times the magnitude of the cross product over the parameter domain. Step 6: Evaluate the integral.

How do you interpret the result of a surface integral over a parallelogram in physical or geometric terms?

The surface integral can represent quantities like flux of a vector field through the surface or the total accumulated value of a scalar function over the surface area, providing geometric or physical insights depending on the context.

Additional Resources

Evaluate The Surface Integral. (x + y + 2) D5, S Is The Parallelogram With Parametric Equations Xu +

Introduction: Understanding Surface Integrals in Multivariable Calculus

In the realm of multivariable calculus, surface integrals serve as powerful tools for quantifying how a vector or scalar field interacts with a surface in three-dimensional space. When dealing with physical phenomena such as heat flow, fluid dynamics, or electromagnetic flux, surface integrals enable us to compute quantities like flux across a surface, total charge passing through a surface, or the amount of a substance diffusing across a boundary.

The specific problem under examination involves evaluating a surface integral of the form:

$$
\iint_{S} (x + y + 2) \, dS
$$

where S is a parallelogram in space defined by parametric equations, and the integrand is a scalar function $f(x, y, z) = x + y + 2$. This scenario exemplifies a scalar surface integral—integrating a scalar function over a surface—often encountered in physics and engineering applications.

The challenge lies in accurately parametrizing the surface, determining the differential surface element dS, and executing the integration. This article provides a comprehensive exploration of this process, starting from fundamental principles and progressing through detailed steps, including the interpretation of parametric equations, computation of surface elements, and the evaluation of the integral.

Understanding the Geometric Context and the Surface S

Defining the Surface: The Parallelogram in Space

The surface S in question is specified as a parallelogram, an important geometric shape in vector calculus due to its simplicity and prevalence. Parallelograms in three dimensions can be described parametrically as the image of a linear transformation of a rectangular domain in the parameter space.

Typically, a parallelogram can be described using two vectors originating from a common point, say $\mathbf{A}$, and spanning the surface:

$$
\mathbf{r}(u, v) = \mathbf{A} + u \mathbf{u} + v \mathbf{v}
$$

where:

- $\mathbf{A}$ is a point on the surface (often the origin or a specific vertex),
- $\mathbf{u}$ and $\mathbf{v}$ are vectors defining the sides of the parallelogram,
- $u, v \in [0, 1]$ (or other bounds depending on the parametrization).

In the problem statement, the surface S is described via parametric equations involving x, y, z, and possibly vectors u, v, although the full explicit equations are incomplete in the prompt. For the purpose of this analysis, we assume the surface is given by a parametric form such as:

$$
\mathbf{r}(u, v) = \mathbf{A} + u \mathbf{u} + v \mathbf{v}
$$

with $u, v \in [0, 1]$. The vectors $\mathbf{u}$ and $\mathbf{v}$ determine the shape and orientation of the parallelogram.

Interpreting the Parametric Equations

Parametric equations are crucial because they translate a complex surface into a manageable domain in the parameter space (u, v). By doing so, the problem of integrating over a surface reduces to a double integral over a parameter rectangle or other domain.

In the context of the problem, the parametric equations involve variables x, y, z expressed in terms of parameters u and v. For example, a typical parametrization might be:

$$
x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)
$$

which maps a rectangle in the (u, v) domain to the surface S. The key is to understand how these equations define the surface's shape and orientation in space, which directly impacts the calculation of dS.

Mathematical Foundations for Surface Integrals

Surface Element dS and Its Computation

To evaluate a surface integral, we need to understand how to compute the differential surface element dS. When a surface S is parametrized by $\mathbf{r}(u, v)$, the surface element is given by:

$$
dS = |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv
$$

where:

- $\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u}$,
- $\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v}$,
- $\times$ denotes the cross product.

This expression encapsulates the infinitesimal area of a parallelogram spanned by the tangent vectors $\mathbf{r}_u$ and $\mathbf{r}_v$.

The magnitude of the cross product, $|\mathbf{r}_u \times \mathbf{r}_v|$, gives the area of this parallelogram. Therefore, the surface integral becomes:

$$
\iint_S f(x, y, z) \, dS = \int_{v_{min}}^{v_{max}} \int_{u_{min}}^{u_{max}} f(\mathbf{r}(u, v)) |\mathbf{r}_u \times \mathbf{r}_v| \, du \, dv
$$

This transformation simplifies the evaluation, provided the parametrization and bounds are clearly specified.

Applying the Surface Integral to the Given Function

The integrand $f(x, y, z) = x + y + 2$ suggests that after parametrization, the integral reduces to integrating:

$$
f(\mathbf{r}(u, v)) = x(u, v) + y(u, v) + 2
$$

over the domain in (u, v), with the integrand multiplied by the magnitude of the cross product of the tangent vectors.

This approach involves:

1. Substituting the parametric equations into f,
2. Calculating $\mathbf{r}_u$ and $\mathbf{r}_v$,
3. Computing the cross product $|\mathbf{r}_u \times \mathbf{r}_v|$,
4. Setting the bounds for u and v,
5. Performing the double integral.

Step-by-Step Evaluation of the Surface Integral

Step 1: Determine the Parametric Equations

Suppose the surface \(S\) is given by:

\[
\mathbf{r}(u, v) = \mathbf{A} + u \mathbf{u} + v \mathbf{v}
\]

where:

- \(\mathbf{A} = (x_0, y_0, z_0)\),
- \(\mathbf{u} = (u_x, u_y, u_z)\),
- \(\mathbf{v} = (v_x, v_y, v_z)\),

and \(u, v \in [0, 1]\). These vectors and points are identified from the problem's initial data or given explicitly.

For example, assume:

\[
\mathbf{A} = (x_0, y_0, z_0),\quad \mathbf{u} = (a, b, c), \quad \mathbf{v} = (d, e, f)
\]

which could be derived from the problem statement once complete.

Step 2: Compute \(\mathbf{r}_u\) and \(\mathbf{r}_v\)

The derivatives are straightforward:

\[
\mathbf{r}_u = \frac{\partial \mathbf{r}}{\partial u} = \mathbf{u} = (a, b, c)
\]

\[
\mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial v} = \mathbf{v} = (d, e, f)
\]

These vectors are constant if the parametrization is linear, simplifying the calculation.

Step 3: Calculate the Cross Product \(|\mathbf{r}_u

\times \mathbf{r}_v|\)

The magnitude of the cross product is computed as:

\[
|\mathbf{r}_u \times \mathbf{r}_v| = | \mathbf{u} \times \mathbf{v} | =
\sqrt{(b f - c e)^2 + (c d - a f)^2 + (a e - b d)^2}
\]

This scalar value represents the area of the parallelogram spanned by
\(\mathbf{u}\) and \(\mathbf{v}\).

Step 4: Express the Integrand \(f(x, y, z)\) in Terms of Parameters

Given the parametrization, the coordinates \(x, y, z\) are functions of \(u, v\):

\[
x(u, v) = x_0 + u a + v d
\]
\[
y(u, v) = y_0

Evaluate The Surface Integral X Y 2 D5 S Is The Parallelogram With Parametric Equations Xu

Evaluate The Surface Integral X Y 2 D5 S Is The Parallelogram With Parametric Equations Xu

Related Articles

- For Each Of The Following Expenses, Select The Best Allocation Basis. A. Value Of Insured Assets. B.
- For This Problem, You May Want To Create A Punnett Square On A Separate Piece Of Paper. Assume Brown
- Currently, 3m Company Pays An Annual Dividend Of $2.46. If The Stock Is Selling For $60, What Is The

Back to Home