Exercise 8.1.2 In Each Case, Write X As The Sum Of A Vector In U And A Vector In U+. A. X=(1, 5, 7),

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Introduction

In linear algebra, the concept of expressing a vector as a sum of vectors belonging to specific subspaces is fundamental. Exercise 8.1.2 provides a practical scenario where, given a vector X = (1, 5, 7), the task is to decompose this vector into the sum of two vectors: one in subspace U and the other in its orthogonal complement U+. This process not only reinforces the understanding of subspace decompositions but also demonstrates the application of projection techniques and orthogonality principles in vector spaces.

In this article, we will explore the steps involved in performing such a decomposition, including defining the subspace U, identifying its orthogonal complement U+, and calculating the respective vectors. We will also discuss the importance of this decomposition in various applications within linear algebra, such as solving systems, understanding projections, and analyzing vector spaces.

Understanding the Problem

What Is Being Asked?

The exercise asks us to:

- Write the vector X = (1, 5, 7) as the sum of two vectors:

$$
\mathbf{X} = \mathbf{u} + \mathbf{v}
$$

where:

- $\mathbf{u} \in U$
- $\mathbf{v} \in U^{\perp}$

- Here, $U^{\perp}$ denotes the orthogonal complement of the subspace U.

Why Is This Important?

Decomposing a vector into components within a subspace and its orthogonal complement is crucial for:

- Simplifying projections
- Solving least squares problems
- Analyzing the structure of vector spaces
- Understanding the geometry of vector decompositions

Assumptions and Prerequisites

Before proceeding, it is essential to specify or deduce the subspace U. Usually, U is defined either explicitly by a basis or by a set of vectors. For this exercise, suppose U is given or can be inferred from context. If the subspace U is not explicitly provided, typical scenarios include:

- U is spanned by a set of vectors
- U is described by certain constraints or equations

Clarification

Since the problem statement in its current form does not specify U, let's assume the following:

- The subspace U is spanned by a given set of vectors, say, for example, $U = \operatorname{span}\{ \mathbf{a}_1, \mathbf{a}_2, \dots \}$

If no such information is provided, the decomposition cannot be uniquely determined. For a comprehensive explanation, we'll proceed with an example where U is spanned by a specific set of vectors.

Defining the Subspace U

Example: Specifying U

Suppose the subspace U is spanned by the vectors:

$$
\mathbf{a}_1 = (1, 0, 0) \quad \text{and} \quad \mathbf{a}_2 = (0, 1, 0)
$$

This choice makes U the xy-plane in $\mathbb{R}^3$.

Implications

- Any vector $\mathbf{u} \in U$ can be written as:

$$
\mathbf{u} = \alpha \mathbf{a}_1 + \beta \mathbf{a}_2 = (\alpha, \beta, 0)
$$

- The orthogonal complement $U^{\perp}$ consists of all vectors orthogonal to every vector in U, which in this case are vectors orthogonal to both $\mathbf{a}_1$ and $\mathbf{a}_2$.

Determining the Orthogonal Complement U+

Finding U+

- Since U is spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$, and these are orthogonal to the z-axis, the orthogonal complement $U^{\perp}$ will be the set of all vectors orthogonal to U, which in this case is:

$$
U^{\perp} = \operatorname{span}\{ (0, 0, 1) \}
$$

- Therefore, any vector $\mathbf{v} \in U^{\perp}$ can be expressed as:

$$
\mathbf{v} = \gamma (0, 0, 1)
$$

Decomposing X into Components in U and U+

Step 1: Express X

Given:

$$
\mathbf{X} = (1, 5, 7)
$$

which we want to write as:

$$
\mathbf{X} = \mathbf{u} + \mathbf{v}
$$

with $\mathbf{u} \in U$, $\mathbf{v} \in U^{\perp}$.

Step 2: Write $\mathbf{u}$ and $\mathbf{v}$

$$
\mathbf{u} = (\alpha, \beta, 0)
$$

$$
\mathbf{v} = (0, 0, \gamma)
$$

Thus:

$$
(1, 5, 7) = (\alpha, \beta, 0) + (0, 0, \gamma)
$$

which implies:

$$
\alpha = 1, \quad \beta = 5, \quad \gamma = 7
$$

Step 3: Confirm the decomposition

- $\mathbf{u} = (1, 5, 0)$
- $\mathbf{v} = (0, 0, 7)$

These satisfy:

$$
\mathbf{X} = \mathbf{u} + \mathbf{v}
$$

and $\mathbf{u} \in U$, $\mathbf{v} \in U^{\perp}$.

General Approach for Other Subspaces

If the subspace U is not as straightforward, the process involves:

1. Identify the basis of U: Find a basis $\{\mathbf{a}_1, \mathbf{a}_2, \dots \}$.

2. Compute the projection of $\mathbf{X}$ onto U:

$$
\operatorname{proj}_U(\mathbf{X}) = \mathbf{U} (\mathbf{U}^T \mathbf{U})^{-1} \mathbf{U}^T

$$
\mathbf{X}
$$

where $\mathbf{U}$ is the matrix with basis vectors as columns.

3. Find the component in $U^{\perp}$:

$$
\mathbf{v} = \mathbf{X} - \operatorname{proj}_U(\mathbf{X})
$$

4. Verify that $\mathbf{v}$ is orthogonal to U: Check that $\mathbf{v}$ is orthogonal to each basis vector of U.

This general method applies to any subspace U, whether it is spanned by one or multiple vectors.

Importance of Orthogonal Decomposition

Orthogonal decomposition of vectors into components within a subspace and its orthogonal complement is a cornerstone of linear algebra because:

- It simplifies complex vector problems by breaking them into manageable parts.
- It underpins the least squares approximation, vital in data fitting.
- It helps in analyzing signals, images, and other data in engineering.
- It provides insight into the structure of vector spaces and their subspaces.

Practical Applications

1. Signal Processing

Decomposing signals into orthogonal components allows filtering and noise reduction.

2. Data Analysis

Principal Component Analysis (PCA) relies on orthogonal projections to reduce dimensionality.

3. Machine Learning

Feature extraction and vector space transformations often involve projecting data onto subspaces.

4. Computer Graphics

Projection techniques are used to render scenes and perform transformations.

Summary

In this detailed exploration of Exercise 8.1.2, we've demonstrated how to decompose a vector into components within a subspace and its orthogonal complement. Using the example where U is spanned by the xy-plane vectors, the decomposition was straightforward, directly splitting the vector into its xy and z components.

In real-world scenarios, the process involves:

- Determining the basis of the subspace.
- Computing the orthogonal projection of the vector onto this subspace.
- Finding the residual vector orthogonal to it.

This approach is fundamental in many areas of mathematics, engineering, and data science, providing a powerful tool for understanding the structure of vector spaces and solving complex problems.

Final Remarks

Understanding how to decompose vectors relative to subspaces enhances intuition in linear algebra and deepens comprehension of the geometric interpretation of vector spaces. Whether working with simple subspaces like coordinate axes or complex subspaces defined by constraints, the principles illustrated in this exercise are universally applicable and form the foundation for advanced linear algebra techniques.

Frequently Asked Questions

What is the goal of Exercise 8.1.2 when given a vector X and subspace U?

The goal is to express the vector X as the sum of a vector in U and a vector in U+, i.e., find vectors u in U and v in U+ such that X = u + v.

How do you determine the vector in U for the given vector X = (1, 5, 7)?

You identify the subspace U and project X onto U to find the vector in U, typically using projection formulas or basis representations depending on U's definition.

What is U+ in the context of Exercise 8.1.2?

U+ represents the orthogonal complement of U within the vector space, containing all vectors orthogonal to every vector in U.

How can the decomposition X = u + v be practically computed given specific U?

Start by finding the projection of X onto U to get u, then subtract u from X to find v, which lies in U+.

Why is it important to write X as a sum of vectors from U and U+?

This decomposition helps understand the structure of the vector space, simplifies computations, and is fundamental in concepts like orthogonal projections and least squares solutions.

Can you provide an example of decomposing X = (1, 5, 7) if U is the subspace spanned by (1, 0, 0) and (0, 1, 0)?

Yes. Project X onto U by taking the components along (1, 0, 0) and (0, 1, 0), which gives u = (1, 5, 0). Then, v = X - u = (0, 0, 7), which lies in U+.

Additional Resources

Exercise 8.1.2: In Each Case, Write X as the Sum of a Vector in U and a Vector in U+

Introduction

In the realm of linear algebra, the concept of decomposing a vector into components relative to a subspace and its orthogonal complement is fundamental. This process not only enhances our understanding of vector spaces but also plays a crucial role in applications such as least squares approximations, projections, and orthogonal decompositions. The exercise under discussion, Exercise 8.1.2, presents a classic scenario: given a vector \(X \), express it as the sum of two vectors—one belonging to a subspace U, and the other belonging to the orthogonal complement $U^{\perp}$.

Specifically, this article examines the problem:

> For the vector $X = (1, 5, 7)$, decompose X as $X = u + v$, where $u \in U$ and $v \in U^{\perp}$.

This task is not merely an exercise in calculation; it embodies the broader principles of orthogonal projections, subspace decomposition, and the geometric intuition that underpins much of linear algebra's power. Through an in-depth exploration, we will analyze the theoretical framework, methods for decomposition, and practical implications, culminating in a comprehensive understanding of the process.

Theoretical Foundations

Subspace Decomposition and Orthogonal Complements

Given a subspace $U \subseteq \mathbb{R}^n$, the entire space $\mathbb{R}^n$ can be viewed as the direct sum of U and its orthogonal complement $U^{\perp}$:

$$
\mathbb{R}^n = U \oplus U^{\perp}
$$

Any vector $X \in \mathbb{R}^n$ can thus be uniquely expressed as:

$$
X = u + v
$$

where:

- $u \in U$
- $v \in U^{\perp}$

This decomposition relies on the orthogonal projection of X onto U, which yields u, and consequently, $v = X - u$.

Orthogonal Projection

The orthogonal projection $P_U(X)$ of a vector X onto U is the vector $u \in U$ such that:

$$
X - u \in U^{\perp}
$$

Mathematically, u minimizes the distance between X and any vector in U:

$$
u = P_U(X)
$$

This projection can be computed explicitly when U is spanned by a set of basis vectors or via matrix methods, especially when U is represented as the column space of a matrix.

Practical Approach to Decomposition

Step 1: Define the Subspace U

The exercise as presented does not specify the subspace U. To proceed, we need to establish U's basis or defining properties. For illustrative purposes, suppose:

$$
U = \text{span}\{ \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k \}
$$

where the basis vectors $\mathbf{a}_i$ are known.

Step 2: Find an Orthonormal Basis for U

Applying the Gram-Schmidt process to the basis vectors of U, we obtain an orthonormal basis $\{ \mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_k \}$. This simplifies the projection calculation:

$$
P_U(X) = \sum_{i=1}^k (X \cdot \mathbf{q}_i) \mathbf{q}_i
$$

where $\cdot$ denotes the dot product.

Step 3: Compute the Projection $u = P_U(X)$

Using the orthonormal basis, compute:

$$
u = P_U(X)
$$

which involves projecting X onto each basis vector and summing.

Step 4: Determine $v = X - u$

Once u is known, the residual $v = X - u$ automatically lies in $U^{\perp}$, completing the decomposition.

Applying the Method to the Specific Vector $X = (1, 5, 7)$

Given that the exercise provides only X, the key step is to specify the subspace U. For the purpose of this review, let's consider a concrete example:

Suppose:

$$
U = \text{span} \left\{ \mathbf{a}_1 = (1, 0, 0), \quad \mathbf{a}_2 = (0, 1, 0) \right\}
$$

This is the xy-plane in $\mathbb{R}^3$.

Explicit Decomposition in the Example

Step 1: Orthonormal Basis

The basis vectors are already orthogonal and of unit length:

$$
\mathbf{q}_1 = (1, 0, 0), \quad \mathbf{q}_2 = (0, 1, 0)
$$

Step 2: Compute the projection u

$$
u = (X \cdot \mathbf{q}_1) \mathbf{q}_1 + (X \cdot \mathbf{q}_2) \mathbf{q}_2
$$

Calculations:

$$
X \cdot \mathbf{q}_1 = (1)(1) + (5)(0) + (7)(0) = 1
$$

$$
X \cdot \mathbf{q}_2 = (1)(0) + (5)(1) + (7)(0) = 5
$$

Thus:

$$
u = 1 \times (1, 0, 0) + 5 \times (0, 1, 0) = (1, 5, 0)
$$

Step 3: Find $v = X - u$

$$
v = (1, 5, 7) - (1, 5, 0) = (0, 0, 7)
$$

Since v is orthogonal to the xy-plane, it belongs to $U^{\perp}$.

Final Decomposition

$$
X = u + v = (1, 5, 0) + (0, 0, 7)
$$

where:

- $u \in U$ (the xy-plane)
- $v \in U^{\perp}$ (the z-axis)

This explicit example demonstrates the process of decomposing a vector into components in a subspace and its orthogonal complement.

Generalization and Broader Implications

Beyond the Example: General Subspaces

While the above illustration is specific, the principles extend to arbitrary subspaces:

- When U is defined by a matrix A with columns spanning U, the projection $P_U(X)$

can be computed via:

$$
P_U(X) = A (A^T A)^{-1} A^T X
$$

- This approach is especially useful in numerical computations, least squares fitting, and signal processing.

Orthogonal Projections in Practice

The ability to decompose vectors is crucial in:

- Data analysis: separating signal components from noise
- Computer graphics: projecting points onto surfaces
- Machine learning: feature extraction and dimensionality reduction
- Statistics: residual analysis in regression

Challenges and Considerations

- Choice of basis: Selecting an orthonormal basis simplifies computations but requires prior steps such as Gram-Schmidt.
- Numerical stability: When dealing with large matrices or ill-conditioned systems, numerical methods must be robust.
- Uniqueness of decomposition: The orthogonal decomposition is unique, contrasting with other decompositions like the QR or LU factorization.

Conclusion

The exercise of decomposing a vector into a sum of vectors in a subspace U and its orthogonal complement $U^{\perp}$ exemplifies a core concept in linear algebra: the orthogonal decomposition theorem. This process not only provides geometric insight into vector spaces but also underpins numerous practical applications across science and engineering.

Using the specific example $X = (1, 5, 7)$ and assuming U as the xy-plane, we demonstrated the step-by-step calculation of the projection, culminating in the explicit decomposition:

$$
X = (1, 5, 0) + (0, 0, 7)
$$

This approach generalizes readily to more complex subspaces and higher-dimensional spaces, emphasizing the versatility and importance of orthogonal projections.

In the broader context, mastering such decompositions equips practitioners with powerful tools for analysis, optimization, and problem-solving in multidimensional settings. Whether in pure mathematics, data science, or applied physics, the ability to dissect vectors into meaningful components remains an essential skill, rooted in the elegant theory of subspace decomposition.

Keywords: linear algebra, orthogonal projection

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