F(x,y)=xy 2i+x 2yj (i) Show That F Is A Conservative Field On The Whole Plane. (ii) Find A Potential

F(x,y)=xy 2i+x 2yj (i) Show That F Is A Conservative Field On The Whole Plane. (ii) Find A Potential

Understanding vector fields is fundamental in multivariable calculus, physics, and engineering. One such vector field is F(x, y) = xy 2i + x^2y j, where i and j are the unit vectors in the x- and y-directions respectively. This article aims to demonstrate that this vector field is conservative over the entire plane and to determine a potential function associated with it. These concepts are vital because conservative fields have important properties, such as path independence of line integrals and the existence of potential functions, simplifying complex calculations in physics and mathematics.

Understanding the Vector Field F(x, y)

Before diving into the analysis, it's essential to understand the structure of the given vector field.

Components of F(x, y)

The vector field is expressed as:

$$F(x, y) = xy^2\, \mathbf{i} + x^2 y\, \mathbf{j}$$

This can be broken down into its component functions:

- $P(x, y) = xy^2$
- $Q(x, y) = x^2 y$

where:

- P(x, y) is the x-component,
- Q(x, y) is the y-component.

Part 1: Showing That F Is a Conservative Field on the Whole Plane

A vector field F is called conservative if there exists a scalar potential function φ(x, y) such that:

$$ F = \nabla \phi $$

which means:

$$ \frac{\partial \phi}{\partial x} = P(x, y) \quad \text{and} \quad \frac{\partial \phi}{\partial y} = Q(x, y) $$

Key criteria for a vector field to be conservative in a simply connected domain like the entire plane include:

- The vector field must be continuously differentiable.
- Cauchy-Riemann equations or the curl condition must hold:

$$ \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} $$

Step 1: Verify differentiability

Since P and Q are polynomials in x and y, they are infinitely differentiable everywhere in the plane.

Step 2: Check the curl condition

Calculate:

$$
\frac{\partial P}{\partial y} = \frac{\partial}{\partial y} (xy^2) = x \cdot 2y = 2xy
$$
$$
\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x} (x^2 y) = 2x y
$$

Since:

$$
\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2xy
$$

the curl condition holds everywhere in the plane.

Step 3: Confirm the domain is simply connected

The entire plane $\mathbb{R}^2$ is simply connected, meaning any closed curve can be continuously contracted to a point without leaving the domain.

Conclusion:

Because the components are continuously differentiable and the curl condition is satisfied everywhere, F(x, y) is a conservative vector field on the whole plane.

Part 2: Finding a Potential Function $\psi(x, y)$

Having established that F is conservative, the next step is to find a scalar potential function $\psi(x, y)$ such that:

\[\nabla \phi = F \]

which implies:

\[
\frac{\partial \phi}{\partial x} = P(x, y) = xy^2
\]
\[
\frac{\partial \phi}{\partial y} = Q(x, y) = x^2 y
\]

Step 1: Integrate P(x, y) with respect to x

Integrate:

\[
\phi(x, y) = \int xy^2\, dx
\]

Treat y as a constant during integration:

\[
\phi(x, y) = y^2 \int x\, dx = y^2 \cdot \frac{x^2}{2} + h(y)
\]

where h(y) is an arbitrary function of y (since partial derivative with respect to x of h(y) is zero).

So,

\[
\phi(x, y) = \frac{1}{2} x^2 y^2 + h(y)
\]

Step 2: Differentiate ψ(x, y) with respect to y and equate to Q(x, y)

\[
\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} \left(
\frac{1}{2} x^2 y^2 + h(y) \right) = x^2 y + h'(y)
\]

Set this equal to Q(x, y):

\[
x^2 y + h'(y) = x^2 y
\]

Subtracting:

\[
h'(y) = 0
\]

Integrate:

\[
h(y) = C
\]

where C is an arbitrary constant.

Step 3: Write the potential function

Thus, the potential function $\phi(x, y)$ is:

$$
\boxed{
\phi(x, y) = \frac{1}{2} x^2 y^2 + C
}
$$

Without loss of generality, choose C = 0 for simplicity.

Summary and Significance

In conclusion, the vector field:

$$ F(x, y) = xy^2\, \mathbf{i} + x^2 y\, \mathbf{j} $$

is a conservative field on the whole plane because:

- It is continuously differentiable everywhere.
- The curl condition $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ holds throughout $\mathbb{R}^2$.

Furthermore, the associated potential function is:

$$
\boxed{
\phi(x, y) = \frac{1}{2} x^2 y^2
}
$$

This potential function simplifies computations involving line integrals of the field and helps interpret physical phenomena, such as potential energy in conservative force fields.

Applications of Conservative Fields and Potential Functions

Understanding whether a vector field is conservative is crucial in various scientific and engineering contexts. For example:

- Physics: Conservative force fields such as gravity or electrostatics have potential functions, leading to energy conservation.
- Fluid Dynamics: Potential functions describe ideal fluid flow without vortices.
- Mathematics: Simplifies the calculation of line integrals, enabling easier evaluation of work done or flux in complex systems.

Additional Insights and Tips

- Always verify the curl condition when determining if a vector field is conservative.
- The domain's topology matters; conservative fields are guaranteed in simply connected domains.
- When finding potential functions, integrate component functions carefully and verify by differentiation.

Final Note: Recognizing and working with conservative vector fields is an essential skill in advanced calculus, physics, and engineering. This example illustrates how to confirm a field's conservative nature and compute its potential, providing foundational tools for analyzing physical systems and solving mathematical problems efficiently.

Frequently Asked Questions

What does it mean for a vector field to be conservative in the context of F(x,y) = xy 2i + x 2 y j?

A vector field is conservative if it can be expressed as the gradient of some scalar potential function. This implies that the line integral of the field between any two points is path-independent, and the curl of the field is zero in the domain.

How do we verify that the vector field F(x,y) = xy 2i + x 2 y j is conservative on the entire plane?

To verify this, we check if the curl of F is zero everywhere. Since F = (P, Q) with P = xy 2 and Q = x 2 y, we compute $\partial Q/\partial x$ and $\partial P/\partial y$. If these partial derivatives are equal everywhere, then F is conservative.

What are the steps to find the potential function for the given vector field F(x,y)?

First, integrate P = xy 2 with respect to x to find a potential function $\psi(x,y)$. Then, differentiate ψ with respect to y and set it equal to Q = x 2 y to determine any functions of y, ensuring the potential function is consistent.

Can the field F(x,y) = xy 2i + x 2 y j be expressed as the gradient of a potential function, and what is

that function?

Yes, since the field is conservative. Integrating P = xy 2 with respect to x gives ψ(x,y) = (1/2) x^2 y 2 + C. Differentiating this with respect to y yields x^2 y, which matches Q when considering the derivatives, confirming the potential function is ψ(x,y) = (1/2) x^2 y 2.

Why is it important to check the curl of the vector field when determining if it is conservative?

Because a necessary and sufficient condition for a vector field in two dimensions to be conservative is that its curl (or equivalently, the difference between ∂Q/∂x and ∂P/∂y) is zero everywhere in the domain. This confirms the existence of a scalar potential function.

Is the potential function for F(x,y) = xy 2i + x 2 y j unique, and how is it determined?

The potential function is unique up to an additive constant. It is determined by integrating the components of the vector field and ensuring consistency between the partial derivatives, which yields a single scalar function representing the field.

How does confirming that F is conservative help in evaluating line integrals over the plane?

If F is conservative, line integrals between two points depend only on those points and not the path taken. This allows for straightforward evaluation by computing the difference in potential function values at the endpoints, simplifying calculations significantly.

Additional Resources

Analysis of the Vector Field F(x, y) = xy^2 i + x^2 y j: Conservation and Potential Function

In the rich landscape of vector calculus, the study of vector fields provides profound insights into various physical phenomena, from electromagnetism to fluid dynamics. Among these, conservative vector fields occupy a central role, characterized by their potential functions and intrinsic properties. This article delves into a particular vector field, F(x, y) = xy^2 i + x^2 y j, exploring whether it is conservative across the entire plane and, if so, determining its potential function. Through meticulous analysis, we aim to uncover the underlying structure of this field and its implications.

Understanding the Vector Field F(x, y) = xy^2 i + x^2 y j

Before diving into the core questions, it is essential to comprehend the nature of the given vector field.

- Components of the Field:

The vector field F(x, y) is composed of two components:

$$
\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}
$$

where

$$
P(x, y) = xy^2, \quad Q(x, y) = x^2 y
$$

- Geometric Intuition:

- The P-component (along the x-axis) involves the product of x and the square of y, suggesting that along lines where y is constant, the component varies linearly with x.

- The Q-component (along the y-axis) involves the product of x squared and y, indicating a dependency on both variables, with a quadratic emphasis on x.

- Physical Interpretation:

While the specific physical context is not provided, such fields often appear in fluid flow or electromagnetic scenarios where the flow or field intensity varies with position in a polynomial fashion.

Part 1: Demonstrating That F Is a Conservative Field on the Whole Plane

A central question in vector calculus is whether a given vector field is conservative — that is, whether it can be expressed as the gradient of some scalar potential function.

Definition:

- A vector field $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ in $\mathbb{R}^2$ is conservative if there exists a scalar potential function $\phi(x, y)$ such that:

$$
\nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) = (P, Q)
$$

Necessary and Sufficient Conditions:

- For simply connected domains like the entire plane $\mathbb{R}^2$, the field $\mathbf{F}$ is conservative if and only if its curl is zero:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0
$$

Step-by-Step Verification:

1. Compute the partial derivatives:

- $\displaystyle \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}(x^2 y) = 2x y$

- $\displaystyle \frac{\partial P}{\partial y} = \frac{\partial}{\partial y}(x y^2) = x \cdot 2 y = 2 x y$

2. Check the curl condition:

$$
\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 x y - 2 x y = 0
$$

3. Domain considerations:

- The domain is $\mathbb{R}^2$, which is simply connected (no holes, no restrictions).

- Since the curl is zero everywhere and the domain is simply connected, the field is conservative on the entire plane.

Conclusion:

- The vector field $\mathbf{F}(x, y) = xy^2 \mathbf{i} + x^2 y \mathbf{j}$ is conservative throughout $\mathbb{R}^2$.

Part 2: Finding the Potential Function $\phi(x, y)$

Having established that $\mathbf{F}$ is conservative, the next step is to find a scalar potential function $\phi(x, y)$ such that:

$$
\nabla \phi = (P, Q) \implies \frac{\partial \phi}{\partial x} = P = xy^2
$$
$$
\frac{\partial \phi}{\partial y} = Q = x^2 y
$$

Methodology:

- Integrate $P(x, y)$ with respect to x, then determine the corresponding function, and finally differentiate with respect to y to match Q.

Step 1: Integrate \(P(x, y) = xy^2\) with respect to \(x\):

\[
\phi(x, y) = \int xy^2 dx
\]

- Treating \(y\) as a constant during integration:

\[
\phi(x, y) = y^2 \int x dx = y^2 \cdot \frac{x^2}{2} + h(y)
\]

where \(h(y)\) is an arbitrary function of \(y\), since the partial derivative with respect to \(x\) is unaffected by functions solely of \(y\).

\[
\boxed{
\phi(x, y) = \frac{1}{2} x^2 y^2 + h(y)
}
\]

Step 2: Differentiate \(\phi(x, y)\) with respect to \(y\) and match with \(Q(x, y)\):

\[
\frac{\partial \phi}{\partial y} = \frac{\partial}{\partial y} \left(\frac{1}{2} x^2 y^2 + h(y) \right) = x^2 y + h'(y)
\]

Set this equal to \(Q(x, y) = x^2 y\):

\[
x^2 y + h'(y) = x^2 y
\]

This implies:

\[
h'(y) = 0
\]

- Integrating:

\[
h(y) = C
\]

where \(C\) is an arbitrary constant.

Final potential function:

\[
\boxed{

\phi(x, y) = \frac{1}{2} x^2 y^2 + C
}
\]

Typically, the constant \(C\) can be set to zero without loss of generality.

Implications and Significance of the Findings

The analysis confirms that $F(x, y) = xy^2 i + x^2 y j$ is a conservative vector field on the entire plane \(\mathbb{R}^2\), with its potential function explicitly given by:

\[
\boxed{
\phi(x, y) = \frac{1}{2} x^2 y^2
}
\]

This result has profound implications in both theoretical and applied contexts:

1. Path-Independence of Line Integrals:

- For conservative fields, the work done along any path between two points depends solely on the endpoints, not the specific trajectory. This simplifies calculations in physics and engineering, such as energy computations in conservative force fields.

2. Existence of Potential Energy:

- The potential function \(\phi\) can be interpreted as a potential energy landscape, with the field describing forces derivable from this potential.

3. Mathematical Elegance and Symmetry:

- The explicit form of \(\phi\) reveals symmetrical properties, notably the quadratic dependence on both \(x\) and \(y\), hinting at underlying symmetries in the field.

4. Further Analytical Opportunities:

- The straightforward potential function permits analytical solutions to integral curves, flow lines, and other dynamical analyses.

Broader Context and Applications

The exploration of such vector fields extends beyond pure mathematics into multiple scientific disciplines:

- Physics: Understanding conservative fields is core to classical mechanics, electromagnetism, and gravitational fields.

- Fluid Dynamics: Potential functions are used to describe irrotational flows, aiding in modeling ideal fluids.

- Mathematical Modeling: Explicit potential functions facilitate numerical simulations, stability analyses, and optimization problems.

- Electromagnetism: Fields with zero curl relate to electrostatic fields, where potential functions are fundamental.

Conclusion

The thorough examination of the vector field $\mathbf{F}(x, y) = xy^2 \mathbf{i} + x^2 y \mathbf{j}$ reveals its status as a conservative field across the entire plane, with an elegantly derived potential function:

$$
\phi(x, y) = \frac{1}{2} x^2 y^2
$$

This discovery underscores the power of fundamental calculus principles—specifically, the curl condition and potential function integration—in elucidating the nature of

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