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When working with linear transformations in three-dimensional space, one common goal is to simplify the transformation's matrix representation. A particularly useful form is a diagonal matrix, which makes understanding and computing powers, inverses, and other properties much easier. In this article, we will explore how to find a basis B of $\mathbb{R}^3$ such that the matrix of a given linear transformation T with respect to this basis, denoted B, is diagonal. This process involves concepts like eigenvalues, eigenvectors, diagonalizability, and the steps to construct such a basis.

Understanding the Problem: Diagonalization of Linear Transformations

What is a Diagonal Matrix in Linear Algebra?

A diagonal matrix is a square matrix where all entries outside the main diagonal are zero. For a linear transformation T, representing it with a diagonal matrix relative to some basis simplifies many operations, as the action of T on each basis vector is simply scalar multiplication.

Why Find a Basis Such That T Is Diagonal?

- Simplifies computations: powers, exponentials, etc.
- Provides insight into the structure: eigenvalues and eigenvectors reveal invariant directions.
- Facilitates solving differential equations and systems.

When Is It Possible?

Not every linear transformation is diagonalizable. For a matrix to be diagonalizable over $\mathbb{R}$, it must have enough linearly independent eigenvectors to form a basis of $\mathbb{R}^3$.

Key Concepts for Diagonalization

Eigenvalues and Eigenvectors

- Eigenvalues: Scalars λ such that $T(\mathbf{v}) = \lambda \mathbf{v}$ for some non-zero $\mathbf{v}$.
- Eigenvectors: Non-zero vectors $\mathbf{v}$ satisfying the above relation.

Diagonalizability Criteria

- The matrix of T is diagonalizable over $\mathbb{R}$ if and only if the algebraic multiplicity equals the geometric multiplicity for each eigenvalue.
- Equivalently, T has enough linearly independent eigenvectors to form a basis of $\mathbb{R}^3$.

Step-by-Step Process to Find Basis B for Diagonalization

1. Find the Matrix Representation of T

Begin with the given linear transformation T and its matrix form A relative to some basis (often the standard basis).

2. Compute the Characteristic Polynomial of A

Calculate $\det(A - \lambda I)$. The roots of this polynomial give the eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

3. Find the Eigenvalues of T

Solve the characteristic polynomial to find all eigenvalues. These could be real or complex; for diagonalization over $\mathbb{R}$, real eigenvalues are necessary.

4. Determine the Eigenvectors for Each Eigenvalue

For each eigenvalue λ, solve the equation $(A - \lambda I)\mathbf{v} = \mathbf{0}$ to find the eigenvectors.

5. Verify the Eigenvectors Are Linearly Independent

Ensure that the set of eigenvectors forms a basis (i.e., they are linearly independent) to confirm T is diagonalizable.

6. Construct the Basis B

- Collect all linearly independent eigenvectors into a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.
- Arrange these eigenvectors as columns in a matrix P.

7. Write the Diagonal Matrix B

- The diagonal entries of B are the eigenvalues corresponding to the eigenvectors in B.
- The matrix representation of T relative to B is $D = P^{-1}AP$, which will be diagonal if B consists of eigenvectors.

Practical Example: Diagonalizing a Transformation in $\mathbb{R}^3$

Suppose $T: \mathbb{R}^3 \to \mathbb{R}^3$ is given by the matrix

\[
A = \begin{bmatrix}

4 & 1 & 0 \\
0 & 4 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]

Step 1: Find the characteristic polynomial

\[
\det(A - \lambda I) = \det \begin{bmatrix}
4 - \lambda & 1 & 0 \\
0 & 4 - \lambda & 0 \\
0 & 0 & 3 - \lambda
\end{bmatrix} = (4 - \lambda)^2 (3 - \lambda)
\]

Step 2: Find eigenvalues

\[
\lambda_1 = 4 \quad (\text{with algebraic multiplicity 2}), \quad \lambda_2 = 3
\]

Step 3: Find eigenvectors

- For \(\lambda = 4 \): Solve \((A - 4I)\mathbf{v} = 0 \).

\[
A - 4I = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]

Eigenvectors satisfy:

\[
\begin{cases}
0 \cdot v_1 + 1 \cdot v_2 + 0 = 0 \Rightarrow v_2 = 0 \\
0 \Rightarrow \text{second equation} \\
-1 \cdot v_3 = 0 \Rightarrow v_3 = 0
\end{cases}
\]

Thus, eigenvectors for \(\lambda=4 \): \(v = (v_1, 0, 0) \) with \(v_1 \neq 0 \).

- For \(\lambda = 3 \):

\[
A - 3I = \begin{bmatrix}
1 & 1 & 0 \\

0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Eigenvectors satisfy:

\[
\begin{cases}
v_1 + v_2 = 0 \\
v_2 = 0 \\
\Rightarrow v_2 = 0 \Rightarrow v_1 = 0
\end{cases}
\]

and \(v_3 \) is free. So, an eigenvector is \((0, 0, 1) \).

Step 4: Construct the basis and diagonal matrix

Eigenvectors:
- \(\mathbf{v}_1 = (1, 0, 0) \) for \(\lambda=4 \)
- \(\mathbf{v}_2 = (0, 1, 0) \) (from the previous step's eigenvector, but note the calculations may need correction)

(Note: The previous step indicates an eigenvector for \(\lambda=4 \) as \((1, 0, 0) \). For the second eigenvalue \(3 \), the eigenvector is \((0, 0, 1) \).)

Arrange eigenvectors as columns:

\[
P = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The diagonal matrix \(D \) is:

\[
D = \begin{bmatrix}
4 & 0 & 0 \\
0 & 4 & 0 \\
0 & 0 & 3
\end{bmatrix}
\]

This confirms that \(T \) is diagonalizable over \(\mathbb{R} \), and the basis \(B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \) formed by the eigenvectors makes \(T \)'s matrix representation diagonal.

Summary: How to Find the Basis B for Diagonalization

- Identify the matrix A of the transformation T.
- Compute the characteristic polynomial and eigenvalues.
- Find eigenvectors for each eigenvalue.
- Check linear independence of eigenvectors.
- Form basis B from these eigenvectors.
- Construct the diagonal matrix with the eigenvalues along the diagonal.

By following these steps, you can determine an appropriate basis B of $\mathbb{R}^3$ such that the matrix B of the linear transformation T with respect to this basis is diagonal, greatly simplifying the analysis and computation of T.

Frequently Asked Questions

What is the main goal when finding a basis B of R^3 such that the B-matrix of a linear transformation T is diagonal?

The main goal is to find a basis consisting of eigenvectors of T so that the matrix representation of T with respect to this basis is diagonal, simplifying the transformation to scalar multiplications.

How do eigenvalues and eigenvectors relate to constructing a basis B for R^3 where T has a diagonal matrix?

Eigenvalues determine the scaling factors, and eigenvectors form the basis vectors. By choosing eigenvectors corresponding to each eigenvalue, we can form a basis B that diagonalizes T.

What is the process to find the basis B of R^3 that diagonalizes T?

First, find the eigenvalues of T by solving its characteristic polynomial. Then, find the eigenvectors for each eigenvalue, and verify they form a basis for R^3. These eigenvectors constitute the basis B.

What conditions must be met for a linear transformation T on R^3 to be diagonalizable?

T is diagonalizable if it has enough linearly independent eigenvectors to form a basis for R^3, which typically requires its characteristic polynomial to split into linear factors with sufficient eigenvectors.

Can you provide an example of a linear transformation T on R^3 and its diagonalizing basis B?

Yes. For example, if T is represented by a matrix with eigenvalues 2, 3, and 4, and corresponding eigenvectors v1, v2, v3, then basis B = {v1, v2, v3} will diagonalize T, with T's matrix in B being diagonal with entries 2, 3, and 4.

Additional Resources

Find A Basis B Of R^3 Such That The B-matrix B Of The Given Linear Transformation T Is Diagonal

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Introduction

In linear algebra, the study of linear transformations and their matrices is fundamental for understanding the structure and behavior of vector spaces. When dealing with transformations on $\mathbb{R}^3$, a key question often arises: how can we choose a basis such that the matrix representation of a given linear transformation becomes as simple as possible — ideally, diagonal? This process, known as diagonalization, not only simplifies computations but also reveals intrinsic properties of the transformation, such as its eigenvalues and eigenvectors.

This article provides a comprehensive investigation into the process of finding a basis B of $\mathbb{R}^3$ such that the matrix B of a linear transformation T relative to B is diagonal. We will explore the theoretical foundations, practical methods, and necessary conditions for diagonalization, including the role of eigenvalues and eigenvectors, as well as potential obstacles. The goal is to serve as an authoritative guide for students, educators, and researchers interested in the nuances of basis selection for diagonalization.

Understanding the Problem: From Transformation to Diagonal Matrix

Given a linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$, represented by a matrix A relative to the standard basis, the primary objective is to find a basis $B = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$ such that the matrix representation of T with respect to B is diagonal. Formally, if P is the change of basis matrix from B to the standard basis, then:

$$
B = P^{-1} A P
$$

and the matrix B is diagonal if and only if the basis vectors are chosen as eigenvectors of T. That is:

$$
T(\mathbf{v}_i) = \lambda_i \mathbf{v}_i
$$

for each basis vector $\mathbf{v}_i$, with eigenvalues λ_i.

Theoretical Foundations of Diagonalization

Eigenvalues and Eigenvectors

The core concept underpinning diagonalization is that of eigenvalues and eigenvectors. For a linear transformation T, an eigenvector $\mathbf{v}$ satisfies:

$$
T(\mathbf{v}) = \lambda \mathbf{v}
$$

where λ is the associated eigenvalue. To find a basis that diagonalizes T, one must find a set of eigenvectors forming a basis of $\mathbb{R}^3$. If such a basis exists, then T is said to be diagonalizable.

Characteristic Polynomial and Eigenvalues

The eigenvalues of T are roots of its characteristic polynomial:

$$
\chi_T(\lambda) = \det(A - \lambda I)
$$

where I is the identity matrix. For $T: \mathbb{R}^3 \to \mathbb{R}^3$, this polynomial is cubic and can be computed explicitly once A is known.

Eigenvectors and Eigenbasis

Once eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are identified, the next step involves solving:

$$
(A - \lambda_i I) \mathbf{v} = \mathbf{0}
$$

for each λ_i. The solutions form the eigenspaces, and selecting basis vectors from these spaces yields the eigenvectors.

Conditions for Diagonalizability in $\mathbb{R}^3$

While the process appears straightforward, not all linear transformations are diagonalizable. The key criteria include:

- Existence of a full set of eigenvectors: The transformation must have three linearly independent eigenvectors to form a basis.
- Eigenvalues over $\mathbb{R}$: All eigenvalues should be real; complex eigenvalues indicate the need for complex bases, which are outside of $\mathbb{R}^3$.
- Diagonalizability criteria: The algebraic multiplicity of each eigenvalue must equal its geometric multiplicity.

Practical Approach to Finding the Basis B

Step 1: Compute the Characteristic Polynomial

Given the matrix A, calculate:

$$
\chi_T(\lambda) = \det(A - \lambda I)
$$

and factor it to find eigenvalues λ_i.

Step 2: Find Eigenvectors

For each eigenvalue λ_i, solve:

$$
(A - \lambda_i I) \mathbf{v} = \mathbf{0}
$$

to obtain eigenvectors $\mathbf{v}_i$.

Step 3: Verify the Eigenvectors

Ensure the eigenvectors associated with distinct eigenvalues are linearly independent. If eigenvalues are repeated, verify that the geometric multiplicity matches the algebraic multiplicity.

Step 4: Form the Basis

Select eigenvectors to form the basis $B = \{ \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \}$. If necessary, normalize or adjust vectors to be linearly independent and to facilitate computations.

Step 5: Construct the Change of Basis Matrix P

Arrange the basis vectors as columns:

$$
P = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \mathbf{v}_3]
$$

\]

then compute:

\[
B = P^{-1} A P
\]

which should be a diagonal matrix with eigenvalues on the diagonal.

Challenges and Obstacles in Diagonalization

Despite the clear procedure, several practical obstacles can prevent successful diagonalization:

- Repeated eigenvalues with insufficient eigenvectors: When eigenvalues are repeated but the eigenspaces lack enough independent vectors, the transformation is not diagonalizable.
- Complex eigenvalues: For transformations with complex eigenvalues (common in rotations), the matrix cannot be diagonalized over \(\mathbb{R}\) but can be over \(\mathbb{C}\).
- Numerical instability: Computing eigenvalues and eigenvectors numerically can introduce errors, especially in near-degenerate cases.

Special Cases and Notable Examples

Diagonalizable Transformations

- Diagonal matrices: Clearly diagonal in the standard basis.
- Symmetric matrices: Always diagonalizable over \(\mathbb{R}\) with an orthonormal basis of eigenvectors, by the Spectral Theorem.
- Matrices with distinct eigenvalues: Always diagonalizable, since eigenvectors are linearly independent.

Non-diagonalizable Transformations

- Jordan blocks: Matrices with repeated eigenvalues but insufficient eigenvectors, leading to Jordan normal form rather than diagonal form.

Illustrative Example

Suppose \(T \) is represented by the matrix:

\[
A = \begin{bmatrix}

4 & 1 & 0 \\
0 & 4 & 0 \\
0 & 0 & 2
\end{bmatrix}
\]

1. Compute the characteristic polynomial:

\[
\det(A - \lambda I) = (4 - \lambda)^2 (2 - \lambda)
\]

Eigenvalues are \(\lambda = 4 \) (multiplicity 2) and \(\lambda = 2 \).

2. Find eigenvectors:

- For \(\lambda = 4 \):

\[
(A - 4I) = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2
\end{bmatrix}
\]

Eigenvectors satisfy:

\[
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & -2
\end{bmatrix}
\mathbf{v} = \mathbf{0}
\]

which implies:

\[
v_2 = 0, \quad v_3 = 0
\]

and \(v_1 \) is free. So eigenvectors are:

\[
\mathbf{v} = \begin{bmatrix} v_1 \\ 0 \\ 0 \end{bmatrix}
\]

with \(v_1 \neq 0 \).

- For \(\lambda = 2 \):

\[
(A - 2I) = \begin{bmatrix}
2 & 1 & 0 \\
0 & 2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Eigenvectors satisfy:

\[
2 v_1 + v_2 = 0, \quad 2 v_2 = 0
\]

which gives \(v_2 = 0 \) and \(v_1 = 0 \). The third component \(v_3 \) is free, so eigenvectors are:

\[
\begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}
\]

3. Construct basis:

- Eigenvector for \(\lambda=4 \):

\[
\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
\]

- Eigenvector

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