Find A Parameterization Of The Line That Is The Intersection Ofthe Planes P: X-2y-z=4 And Q:2x+y+z=2

Find A Parameterization Of The Line That Is The Intersection Of the Planes P: X - 2y - z = 4 And Q: 2x + y + z = 2

Understanding how to find the line of intersection between two planes is fundamental in analytic geometry. When two planes intersect, they do so along a line, and finding a parameterization of this line is essential for various applications, including vector calculus, 3D modeling, and geometric problem-solving. This guide provides a comprehensive, step-by-step approach to determining the parametric equations of the intersection line for the planes P and Q defined as:

- Plane P: $x - 2y - z = 4$
- Plane Q: $2x + y + z = 2$

By the end of this article, you will understand how to systematically find the parametric equations representing the line where these two planes intersect, including solving the system, choosing parameters, and verifying the solution.

Understanding the Problem: Intersection of Two Planes

The intersection of two planes in three-dimensional space results in a straight line, provided the planes are not parallel or coincident. To find this line:

- Identify the planes involved: In this case, P and Q.
- Express the planes in standard form: Both are already in standard form.
- Solve the system of equations: The intersection line satisfies both equations simultaneously.

The goal: Find a set of parametric equations $(x(t), y(t), z(t))$ that describe all points lying on both planes.

Step 1: Write Down the Equations of the Planes

The given planes are:

- Plane P: $x - 2y - z = 4$
- Plane Q: $2x + y + z = 2$

Expressed as a system:

$$
\begin{cases}
x - 2y - z = 4 \quad (1) \\
2x + y + z = 2 \quad (2)
\end{cases}
$$

Step 2: Solve the System of Equations

To find the intersection line, we solve equations (1) and (2) simultaneously.

Method:

- Use substitution or elimination methods.
- Choose a variable to parameterize, often y or z.

Step 2.1: Express variables in terms of a parameter

Let's choose y as a free parameter:

- Assign $y = t$, where $t \in \mathbb{R}$.

Step 2.2: Solve for x and z in terms of t

From equation (1):

$$
x - 2t - z = 4 \Rightarrow x = 4 + 2t + z
$$

From equation (2):

$$
2x + t + z = 2
$$

Substitute x from above:

$$
2(4 + 2t + z) + t + z = 2
$$

Simplify:

$$
8 + 4t + 2z + t + z = 2
$$

Combine like terms:

$$
8 + 5t + 3z = 2
$$

Solve for z:

$$
3z = 2 - 8 - 5t = -6 - 5t
\Rightarrow z = -2 - \frac{5}{3}t
$$

Now, substitute z back into the expression for x:

$$
x = 4 + 2t + z = 4 + 2t + \left(-2 - \frac{5}{3}t \right)
$$

Simplify:

$$
x = (4 - 2) + 2t - \frac{5}{3} t = 2 + 2t - \frac{5}{3} t
$$

Express \(2t \) as \(\frac{6}{3} t \):

\[
x = 2 + \left(\frac{6}{3} t - \frac{5}{3} t \right) = 2 + \frac{1}{3} t
\]

Summary of parametric expressions:

\[
\boxed{
\begin{cases}
x(t) = 2 + \frac{1}{3} t \\
y(t) = t \\
z(t) = -2 - \frac{5}{3} t
\end{cases}
}
\]

where \(t \in \mathbb{R} \).

Step 3: Write the Parameterization of the Intersection Line

The parametric equations derived are:

\[
\begin{aligned}
x &= 2 + \frac{1}{3} t \\
y &= t \\
z &= -2 - \frac{5}{3} t
\end{aligned}
\]

This can be expressed in vector form as:

\[
\mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{v}
\]

where:

- $\mathbf{r}_0$ is a point on the line.
- $\mathbf{v}$ is the direction vector of the line.

Identify a point on the line:

Choose $t = 0$:

$$
x(0) = 2, \quad y(0) = 0, \quad z(0) = -2
$$

So, a point on the line is:

$$
\mathbf{r}_0 = (2, 0, -2)
$$

Determine the direction vector $\mathbf{v}$:

From the equations, the coefficients of t give:

$$
\mathbf{v} = \left(\frac{1}{3}, 1, -\frac{5}{3} \right)
$$

To simplify, multiply $\mathbf{v}$ by 3 to clear denominators:

$$
\mathbf{v} = (1, 3, -5)
$$

Thus, the parametric equations can be rewritten as:

$$
\boxed{
\mathbf{r}(t) = (2, 0, -2) + t (1, 3, -5), \quad t \in \mathbb{R}
}
$$

which explicitly describes the line of intersection.

Step 4: Verify the Solution

To ensure correctness, verify that the point and direction satisfy both plane equations.

Check point $\mathbf{r}_0 = (2, 0, -2)$:

- For plane P:

$$
x - 2y - z = 2 - 0 - (-2) = 2 + 2 = 4 \quad \checkmark
$$

- For plane Q:

$$
2x + y + z = 2(2) + 0 + (-2) = 4 - 2 = 2 \quad \checkmark
$$

Check the direction vector:

- The vector $\mathbf{v} = (1, 3, -5)$ should be parallel or consistent with the planes' normals, as the line is their intersection.

Normal vectors:

- For plane P: $\mathbf{n}_P = (1, -2, -1)$
- For plane Q: $\mathbf{n}_Q = (2, 1, 1)$

Dot products with $\mathbf{v}$:

$$
\mathbf{n}_P \cdot \mathbf{v} = 1 \times 1 + (-2) \times 3 + (-1) \times (-5) = 1 - 6 + 5 = 0
$$
$$
\mathbf{n}_Q \cdot \mathbf{v} = 2 \times 1 + 1 \times 3 + 1 \times (-5) = 2 + 3 - 5 = 0
$$

Both dot products are zero, confirming that the direction vector is orthogonal to the normals, consistent with the line lying in both planes.

Additional Insights: Geometric Interpretation

Understanding the geometric context enhances comprehension:

- The direction vector indicates the line's orientation in space.
- The point \(\mathbf{r}_0\) is a specific solution satisfying both plane equations.
- The parametric form provides a complete description, allowing for plotting, calculations, and further analysis.

Applications of the Parameterized Line

Once the line of intersection has been established, numerous applications follow:

- Distance calculations: Find the shortest distance from a point to the line.
- Intersection with other geometries: Check for intersections with other geometric entities.
- Vector calculus operations: Compute line integrals or work with vector fields along the line.
- 3D modeling: Use in CAD software for designing complex objects.

Conclusion

In this comprehensive guide, we've demonstrated how to find a parameterization of the line of intersection between two planes:

- Started with the system of equations.
- Chose a free parameter to simplify the solution.
- Solved systematically for all variables.
- Expressed the

Frequently Asked Questions

How do I find the line of intersection between the planes P: $x - 2y - z = 4$ and Q: $2x + y + z = 2$?

To find the intersection line, solve the system of equations formed by the two planes. You can express one variable in terms of others or set a parameter and find parametric equations for the line.

What is the first step in parameterizing the intersection line of two planes?

Choose a variable to parameterize (e.g., set $y = t$), then substitute into the equations to solve for x and z in terms of t.

How do I solve for the parametric equations of the line from the plane equations?

Solve one of the equations for one variable in terms of the parameter, then substitute into the other to find expressions for the remaining variables, resulting in parametric equations.

Can you provide a step-by-step solution to find the line of intersection for the given planes?

Yes. For example, set $y = t$. Substitute into the planes: from P: $x - 2t - z = 4 \Rightarrow x = 4 + 2t + z$. From Q: $2x + t + z = 2$. Substitute x: $2(4 + 2t + z) + t + z = 2$. Simplify and solve for z, then find x accordingly.

What are the parametric equations for the intersection line of the planes $x - 2y - z = 4$ and $2x + y + z = 2$?

Set $y = t$, then solve for z and x: $z = -t + 2$, $x = 4 + 2t + z = 4 + 2t + (-t + 2) = 6 + t$. Therefore, the parametric equations are: $x = 6 + t$, $y = t$, $z = -t + 2$.

What is the vector form of the intersection line between the two planes?

The line can be expressed as $r(t) = (6, 0, 2) + t(1, 1, -1)$, where $(6, 0, 2)$ is a point on the line and $(1, 1, -1)$ is the direction vector.

How do I verify that my parameterization correctly represents the intersection line?

Substitute the parametric equations back into both plane equations to ensure they satisfy both equations for all values of t.

What is the importance of choosing a variable to parameterize in this problem?

Choosing a variable simplifies solving the system because it allows expressing the other variables in terms of the parameter, making the line's equations easier to derive.

Are there alternative methods to find the intersection line besides substitution?

Yes, methods include using the cross product of the planes' normal vectors to find the direction vector and solving for a specific point on the line, then formulating the parametric equations.

What is the geometric interpretation of the line formed by the intersection of two planes?

The intersection line is the set of all points that lie on both planes simultaneously, representing a line where the two planes meet in three-dimensional space.

Additional Resources

Finding a Parameterization of the Line of Intersection of Two Planes: A Comprehensive Guide

Understanding the intersection of planes in three-dimensional space is a fundamental aspect of analytical geometry. When two planes intersect, they do so along a line, and finding a parameterization of this line offers valuable insights into spatial relationships, vector equations, and geometric intuition. This guide delves deeply into the process of determining the parameterization of the intersection line of the planes:

- $P: x - 2y - z = 4$
- $Q: 2x + y + z = 2$

We will systematically explore the problem, starting with the foundational concepts, moving through algebraic methods, and concluding with the explicit parameterized form of the intersection line.

Understanding the Planes and Their Intersection

The Geometric Nature of Planes in 3D Space

- Each plane in three-dimensional space can be represented by a linear equation involving three variables x, y, z.
- The set of all points satisfying the plane equation forms a flat, two-dimensional surface.
- When two planes are not parallel and are not coincident, they intersect along a line—an infinite set of points satisfying both equations simultaneously.

Why Find a Parameterization?

- A parameterization expresses the set of points on the intersection line as a function of a single parameter, often denoted t.
- It provides a convenient way to describe the line algebraically and geometrically, facilitating calculations such as distance, angle, and points along the line.
- It is essential in applications like computer graphics, physics, and engineering, where precise descriptions of spatial relationships are crucial.

Mathematical Approach to Find the Intersection Line

Step 1: Write the System of Equations

Given the planes:

$$
\begin{cases}
x - 2y - z = 4 \quad (1) \\
2x + y + z = 2 \quad (2)
\end{cases}
$$

This system describes all points (x, y, z) lying on both planes.

Step 2: Express the System in a Suitable Form

- The goal is to find expressions for x, y, z in terms of a free parameter.
- Since there are two equations with three unknowns, we expect at least one variable to be free.

Step 3: Use Algebraic Methods to Solve

- The typical approach involves solving for two variables in terms of the third, which will serve as the parameter.

Solving the System for a Parameterization

Step 4: Choose a Variable to Parameterize

- It is often easiest to pick one variable as the free parameter.
- Let's choose $y = t$, where t is a real parameter.

Step 5: Solve for x and z in Terms of t

- Substitute $y = t$ into the two equations:

$$
\begin{cases}
x - 2t - z = 4 \quad (1a) \\
2x + t + z = 2 \quad (2a)
\end{cases}
$$

- From equation (1a):

$$
x = 4 + 2t + z
$$

- From equation (2a):

$$
2x + t + z = 2
$$
$$
2(4 + 2t + z) + t + z = 2
$$
$$
8 + 4t + 2z + t + z = 2
$$
$$
8 + 5t + 3z = 2
$$

- Solve for z:

$$
3z = 2 - 8 - 5t
$$
$$
3z = -6 - 5t
$$
$$
z = -2 - \frac{5}{3} t
$$

- Now, substitute z back into the expression for x:

$$
x = 4 + 2t + z = 4 + 2t - 2 - \frac{5}{3} t
$$
$$
x = (4 - 2) + 2t - \frac{5}{3} t = 2 + 2t - \frac{5}{3} t
$$

- Combine the t-terms:

$$
2t - \frac{5}{3} t = \left(2 - \frac{5}{3} \right) t = \left(\frac{6}{3} - \frac{5}{3} \right) t = \frac{1}{3} t
$$

- Therefore,

$$
x = 2 + \frac{1}{3} t
$$

Step 6: Write the Complete Parameterization

- Summarizing the results:

$$
\boxed{
\begin{cases}
x = 2 + \frac{1}{3} t \\
y = t \\
z = -2 - \frac{5}{3} t
\end{cases}
}
$$

- The parameter t varies over all real numbers, covering every point on the line of intersection.

Vector Form of the Line of Intersection

Expressing as a Vector Equation

- The parametric equations can be written in vector form:

$$
\mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{d}
$$

where:

- $\mathbf{r}_0$ is a point on the line (when $t=0$)
- $\mathbf{d}$ is the direction vector of the line

Identifying a Point on the Line

- When $t=0$:

$$
x=2, \quad y=0, \quad z=-2
$$

- So, a point on the line is:

$$
\mathbf{r}_0 = (2, 0, -2)
$$

Determining the Direction Vector

- From the parametric equations, the coefficients of t give the direction vector:

$$
\mathbf{d} = \left(\frac{1}{3}, 1, -\frac{5}{3} \right)
$$

- To avoid fractions, multiply the vector by 3:

$$
\mathbf{d} = (1, 3, -5)
$$

- The direction vector can be taken as $\mathbf{d} = (1, 3, -5)$.

Final Vector Equation of the Line

$$
\boxed{
\mathbf{r}(t) = (2, 0, -2) + t (1, 3, -5), \quad t \in \mathbb{R}
}
$$

This concise form allows easy computation of points along the line for any value of t.

Verification and Consistency Checks

Substitute Back into the Plane Equations

- To confirm the parameterization is correct, substitute the parametric coordinates into both plane equations:

1. $x - 2 y - z = 4$:

$$
(2 + \frac{1}{3} t) - 2 (t) - \left(-2 - \frac{5}{3} t \right) = 4
$$

Simplify:

$$
2 + \frac{1}{3} t - 2 t + 2 + \frac{5}{3} t = 4
$$

Combine like terms:

$$
(2 + 2) + \left(\frac{1}{3} t - 2 t + \frac{5}{3} t \right) = 4
$$

$$
4 + \left(\frac{1}{3} t + \frac{5}{3} t - 2 t \right) = 4
$$

Express $-2 t$ as $- \frac{6}{3} t$:

$$
4 + \left(\frac{1}{3} t + \frac{5}{3} t - \frac{6}{3} t \right) = 4
$$

$$
4 + \left(\frac{1 + 5 - 6}{3} t \right) = 4
$$

$$
4 + 0 \cdot t = 4
$$

True for all \(t \).

2. \(2x + y + z = 2 \):

\[
2 (2 + \frac{1}{3} t) + t + \left(-2 - \frac{5}{3} t \right) = 2
\]

Simplify:

\[
4 + \frac{2}{3} t + t - 2 - \frac{5}{3} t = 2
\]

Combine constants:

\[
(4 - 2) + \left(\frac{2}{3} t + t - \frac{5}{3

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Find A Parameterization Of The Line That Is The Intersection Ofthe Planes P X 2y Z 4 And Q 2x Y Z 2

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