Find A Polar Equation Of The Conic With Its Focus At The Pole.Conic: ParabolaVertex Or Vertices: (1,

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Understanding conic sections is fundamental in the study of geometry and algebra. Among the various types of conic sections—circles, ellipses, parabolas, and hyperbolas—the parabola holds particular significance due to its applications in physics, engineering, and astronomy. One common problem involves deriving the polar equation of a conic with its focus at the pole (origin). This article provides a comprehensive guide to finding the polar equation of a parabola when its focus is at the pole, specifically focusing on the case where the vertex or vertices are at the point (1, 0).

Introduction to Conic Sections in Polar Coordinates

Conic sections can be expressed elegantly in polar coordinates, which are centered at a focus point rather than the origin of the Cartesian plane. In polar coordinates:

- The pole (origin) is the focus of the conic.
- The radius vector r extends from the focus to a point on the conic.
- The angle θ is measured from a fixed reference line, often the positive x-axis.

Expressing conics in polar form allows for a more straightforward analysis of their properties, especially when the focus is at the pole. The general polar equation for a conic with focus at the pole is:

$$
r = \frac{e d}{1 + e \cos \theta}
$$

where:
- e is the eccentricity,
- d is the distance from the focus to the directrix.

However, for a parabola, this equation simplifies because the eccentricity $e = 1$.

The Parabola in Polar Coordinates with Focus at the Pole

Standard Equation of a Parabola

The parabola is defined as the locus of points equidistant from a fixed point called the focus and a fixed line called the directrix. When the focus is at the pole and the directrix is a vertical line, the parabola's equation takes a specific form in polar coordinates.

Equation of a Parabola with Focus at the Origin

When the focus is at the pole (origin), the parabola's equation can be expressed as:

$$
r = \frac{l}{1 + \cos \theta}
$$

or

$$
r = \frac{l}{1 - \cos \theta}
$$

depending on the orientation of the parabola.

- l is the semi-latus rectum, the distance from the focus to the parabola along the line perpendicular to the axis of symmetry.

Deriving the Equation from Vertex or Vertices at (1, 0)

If the parabola has vertices at (1, 0), it indicates that the parabola is positioned such that its vertex lies at this point, and the focus is at the origin. To find the polar equation in this case, we need to analyze the parabola's geometric properties and relate them to the polar form.

Step-by-Step Approach to Find the Polar Equation

Step 1: Identify the Focus and Vertex Positions

- Focus at the origin $(0, 0)$.
- Vertex at $(1, 0)$.

Since the vertex is at $(1, 0)$, and the focus is at $(0, 0)$, the parabola opens to the right along the x-axis.

Step 2: Determine the Parameters

- Distance from focus to vertex $= |1 - 0| = 1$.
- For a parabola, the vertex lies midway between the focus and directrix.
- The distance from the focus to the vertex a is $a = 1$.

The directrix is located at a perpendicular distance a from the vertex, on the opposite side of the focus.

Step 3: Write the Equation in Cartesian Coordinates

The parabola with focus at $(0, 0)$ and vertex at $(1, 0)$ can be written as:

$$
(x - h)^2 = 4p(y - k)
$$

or, for the parabola opening along the x-axis:

\[

(y - k)^2 = 4p(x - h)
\]

Given the focus at \((0, 0) \) and vertex at \((1, 0) \), the parabola opens to the right (positive x). The standard form becomes:

\[
(y - 0)^2 = 4p(x - 0)
\]

Since the vertex is at \((1, 0) \), and for a parabola opening right, the focus is at \((p, 0) \). But here, the focus is at \((0, 0) \), and the vertex is at \((1, 0) \).

Using the focus-directrix property:

- The distance from vertex to focus \(= 1 \).
- The parabola's parameter \(p \) is the distance from the vertex to the focus, which is \(p = -1 \) if the parabola opens left, or \(p = 1 \) if it opens right.

Since the focus is at \((0, 0) \), and the vertex at \((1, 0) \), the parabola opens to the left (towards decreasing x), with \(p = -1 \).

Thus, the parabola's Cartesian equation is:

\[
(y)^2 = -4(x - 1)
\]

or equivalently:

\[
x = 1 - \frac{y^2}{4}
\]

Step 4: Convert the Cartesian Equation to Polar Coordinates

Recall the polar to Cartesian conversions:

\[
x = r \cos \theta
\]
\[
y = r \sin \theta
\]

Substituting into the Cartesian equation:

\[
r \cos \theta = 1 - \frac{(r \sin \theta)^2}{4}
\]

which simplifies to:

\[
r \cos \theta = 1 - \frac{r^2 \sin^2 \theta}{4}
\]

Rearranged:

\[
r^2 \sin^2 \theta + 4 r \cos \theta - 4 = 0
\]

This is a quadratic in \(r \):

\[
r^2 \sin^2 \theta + 4 r \cos \theta - 4 = 0
\]

Step 5: Solve for \(r \)

Applying the quadratic formula:

\[
r = \frac{-4 \cos \theta \pm \sqrt{(4 \cos \theta)^2 - 4 \sin^2 \theta (-4)}}{2 \sin^2 \theta}
\]

Compute discriminant:

\[
(4 \cos \theta)^2 - 4 \sin^2 \theta (-4) = 16 \cos^2 \theta + 16 \sin^2 \theta = 16 (\cos^2 \theta + \sin^2 \theta) = 16
\]

Thus,

\[
r = \frac{-4 \cos \theta \pm \sqrt{16}}{2 \sin^2 \theta} = \frac{-4 \cos \theta \pm 4}{2 \sin^2 \theta}
\]

Simplify numerator:

\[
r = \frac{-4 \cos \theta \pm 4}{2 \sin^2 \theta}
\]

Break into two solutions:

1. Positive root:

\[

r = \frac{-4 \cos \theta + 4}{2 \sin^2 \theta} = \frac{4(1 - \cos \theta)}{2 \sin^2 \theta} = \frac{2(1 - \cos \theta)}{\sin^2 \theta}
\]

2. Negative root:

\[
r = \frac{-4 \cos \theta - 4}{2 \sin^2 \theta} = \frac{-4(\cos \theta + 1)}{2 \sin^2 \theta} = -\frac{2(\cos \theta + 1)}{\sin^2 \theta}
\]

Since \(r \) represents a distance, it must be positive, so we select the positive solution:

\[
r = \frac{2(1 - \cos \theta)}{\sin^2 \theta}
\]

Recall the trigonometric identity:

\[
1 - \cos \theta = 2 \sin^2 \frac{\theta}{2}
\]

and

\[
\sin^2 \theta = 4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}
\]

Substitute into the equation:

\[
r = \frac{2 \times 2 \sin^2 \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} = \frac{4 \sin^2 \frac{\theta}{2}}{4 \sin^2 \frac{\theta}{2} \cos^2 \frac{\theta}{2}} = \frac{1}{\cos^2 \frac{\theta}{2}}
\]

Thus,

\[
r = \frac{1}{\cos^2 \frac{\theta}{2}}
\]

which can

Frequently Asked Questions

What is the general form of the polar equation of a parabola with its focus at the pole?

The general form is $r = e d / (1 + e \cos \theta)$, where e is the eccentricity (for a parabola $e = 1$), and d is the distance from the focus to the vertex.

Given the focus at the pole and vertex at (1, 0), how do you find the polar equation of the parabola?

Since the vertex is at (1, 0) and the parabola has its focus at the pole, the distance from focus to vertex is 1. The parabola's polar equation simplifies to $r = 2\ 1 / (1 + \cos \theta)$, or $r = 2 / (1 + \cos \theta)$.

What is the significance of the eccentricity e = 1 in the polar equation of a parabola?

An eccentricity of $e = 1$ indicates the conic is a parabola, which has a focus at the pole and a directrix perpendicular to the axis of symmetry.

How does the vertex at (1, 0) influence the form of the polar equation for the parabola?

The vertex at (1, 0) suggests the parabola opens along the positive x-axis, and the equation can be expressed as $r = 2 / (1 + \cos \theta)$, with the vertex at the point where $\theta = 0$.

Can the polar equation of this parabola be written in terms of the given vertex coordinate (1, 0)?

Yes, since the vertex is at (1, 0), the polar equation with focus at the pole becomes $r = 2 / (1 + \cos \theta)$, where the vertex corresponds to $\theta = 0$ and $r = 1$.

What is the role of the directrix in deriving the polar equation of the parabola with focus at the pole?

The directrix is used to define the parabola: points are equidistant from the focus (pole) and the directrix. This geometric property helps derive the polar form, leading to $r = 2a / (1 + \cos \theta)$ for a parabola with vertex at (a, 0).

If the parabola's vertex is at (1, 0) and focus at the pole, what is the directrix's position?

The directrix is located at $x = -1$, equidistant from the focus at (0, 0) and the vertex at (1, 0), since the vertex is midway between focus and directrix.

How do you verify that the polar equation r = 2 / (1 + cos θ)

corresponds to a parabola with focus at the pole and vertex at (1, 0)?

By plotting the equation or analyzing points at specific angles (e.g., $\theta = 0$, $\pi/2$), you can confirm the focus at the pole and that the vertex occurs at $r = 1$ when $\theta = 0$, matching the vertex at (1, 0).

What are common applications of finding the polar equation of a parabola with its focus at the pole?

This form is used in astronomy, satellite dish design, and optics, where parabolic shapes with a focus at the pole simplify calculations of reflection, focus, and path trajectories.

Additional Resources

Find A Polar Equation Of The Conic With Its Focus At The Pole. Conic: Parabola Vertex Or Vertices: (1,)

Understanding the geometric and algebraic properties of conic sections is fundamental in advanced mathematics, physics, and engineering applications. Among the various forms of conic sections—parabolas, ellipses, and hyperbolas—their representation in polar coordinates offers both elegance and analytical power, especially when the focus or a key point of the conic is conveniently positioned at the pole (origin). This article delves into the process of deriving a polar equation of a conic with its focus at the pole, with particular emphasis on the parabola, and explores the implications of given vertices, such as (1,), in determining the conic’s parameters and form.

Understanding Conic Sections in Polar Coordinates

Basics of Polar Coordinates

Before exploring conic sections, it is essential to understand the polar coordinate system. Unlike Cartesian coordinates (x, y), polar coordinates (r, θ) define a point’s location based on its distance from the origin (pole) and the angle from a fixed reference direction (usually the positive x-axis). This system is particularly advantageous in dealing with curves that have symmetry about a point, such as conics.

- r (radius): Distance from the pole to the point.
- θ (theta): Angle between the positive x-axis and the line segment from the pole to the point.

Conic Sections in Polar Form

A conic section can be represented in polar coordinates with a focus at the pole by the general equation:

$$ r = \frac{e d}{1 + e \cos \theta} $$

or

$$ r = \frac{e d}{1 + e \sin \theta} $$

where:

- e is the eccentricity (a measure of how much the conic deviates from being circular),
- d is the distance from the focus to the directrix,
- The choice between cosine or sine depends on the conic’s orientation and the position of the focus relative to the axes.

For conics with the focus at the pole:

- When $0 < e < 1$, the conic is an ellipse.
- When $e = 1$, it’s a parabola.
- When $e > 1$, it’s a hyperbola.

This framework simplifies the analysis and helps derive specific equations when the focus is at the pole, which is common in orbital mechanics and physics.

Focus at the Pole: Significance and Geometric Implications

Placing the focus at the pole simplifies many calculations and aligns with the physical interpretations of conic sections, especially in planetary orbits and optics. The focus being at the origin (pole) means the conic’s key point, which directs the shape and properties, coincides with the coordinate origin.

Implications:

- The vertex of the parabola or the vertices of the ellipse can be described relative to the focus.
- Equations become more straightforward and symmetric.
- The conic’s symmetry axis aligns with either the x-axis or y-axis depending on its orientation.

Deriving the Polar Equation of a Parabola with Focus at the Pole

Properties of a Parabola in Polar Coordinates

A parabola is characterized by its eccentricity $e = 1$. When the focus is at the pole, the standard form of the parabola in polar coordinates simplifies to:

$$r = \frac{2 p}{1 + \cos \theta}$$

where:

- p is the distance from the focus to the directrix, which also equals the distance from the focus to the vertex along the axis of symmetry.

Key features:

- The parabola opens along the axis where the directrix is perpendicular.
- The vertex is located at $r = p$, at $\theta = 0$ or π, depending on orientation.

Step-by-step Derivation

1. Identify the focus at the pole: The focus is at the origin (0,0).
2. Determine the directrix: For a parabola with focus at the origin and directrix perpendicular to the axis of symmetry, the directrix can be expressed as a line $x = -p$ or $y = p$, depending on orientation.
3. Use the definition of a parabola: The set of points equidistant from the focus and the directrix.

Mathematically:

$$r = \text{distance from focus}$$
$$\text{Distance from directrix} = \left| r \cos \theta + p \right|$$

The condition:

$$r = \left| r \cos \theta + p \right|$$

leads to:

$$r = r \cos \theta + p$$

assuming the parabola opens to the right.

Rearranged:

$$r (1 - \cos \theta) = p$$

which simplifies to:

$$r = \frac{p}{1 - \cos \theta}$$

Alternatively, by considering the standard form, we arrive at:

$$ r = \frac{2 p}{1 + \cos \theta} $$

for a parabola opening to the right, with the focus at the pole.

Vertex of the Parabola

The vertex is located midway between the focus and the directrix, at a point where r reaches its minimum:

$$ r_{min} = p $$

and at $\theta = 0$, the equation yields:

$$ r = \frac{2 p}{1 + 1} = p $$

confirming the vertex's position at $r = p$, along the axis of symmetry.

Incorporating Vertices and Additional Points: (1,)

The mention of vertices such as (1,) suggests a specific positional information about the conic. Vertices are points on the conic where the curvature is extremal, often the closest or farthest points from the focus, and are crucial in defining the size and shape of the conic.

Analyzing given vertices:

- If a vertex is at (1,), depending on the coordinate system, it could specify the x-coordinate or y-coordinate.
- For a conic with focus at the pole, the vertex’s position relative to the focus provides key data for the equation.

Methodology:

1. Identify the position of the vertex: For example, if the vertex is at (1, 0), and the focus is at the origin, then the vertex is 1 unit along the x-axis.
2. Determine the corresponding eccentricity and parameters: The location of the vertex relative to the focus influences the conic’s parameters, especially for the parabola where the vertex lies midway between the focus and the directrix.
3. Use the vertex to find the parameter p: Since for a parabola, the vertex is at $r = p$ when $\theta = 0$ (or π), the distance from focus to vertex directly relates to p.

Constructing the Polar Equation from Vertex and Focus Data

Suppose the focus is at the pole (0,0), and the vertex is at (1,0). For a parabola:

- The vertex lies midway between the focus and directrix.
- The directrix is a line perpendicular to the axis of symmetry.

Given:

- The focus at (0,0).
- The vertex at (1,0).

This suggests that the directrix is located at a point such that the distance from the focus to the vertex is equal to the distance from the vertex to the directrix.

Since the focus is at the pole and the vertex is 1 unit along the x-axis:

- The distance from focus to vertex: 1.
- The directrix should then be at a point 1 unit away from the vertex, on the opposite side of the focus.

So, the directrix is at $x = 2$, assuming the parabola opens rightward.

The parabola’s equation in polar form becomes:

$$r = \frac{2 p}{1 + \cos \theta}$$

where $p = 1$ (distance from focus to vertex). Substituting $p = 1$:

$$r = \frac{2 \times 1}{1 + \cos \theta} = \frac{2}{1 + \cos \theta}$$

which aligns with the earlier derivation.

Analyzing and Interpreting the Equation

Once the polar equation is established, several analyses can be performed:

- Plotting the conic: The graph of $r = \frac{2}{1 + \cos \theta}$ produces a parabola opening to the right, with the focus at the pole.
- Vertex location: When $\theta = 0$:

$$r = \frac{2}{1 + 1} = 1$$

which matches the vertex at (1, 0).

- Symmetry: The equation is symmetric about the axis $\theta = 0$.

- Behavior at various angles:

- At $\theta = \pi$:

$$ r = \frac{2}{1 - 1} \rightarrow \infty $$

indicating

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