Find A Power Series Expansions For 1/(1-x)^2, 1/(1-x)^3, 1/(1-x)^4, By Differentiating 1/(1-x). Give

Find A Power Series Expansions For 1/(1-x)^2, 1/(1-x)^3, 1/(1-x)^4, By Differentiating 1/(1-x). Give

Understanding power series expansions is a fundamental aspect of calculus and mathematical analysis, especially when dealing with functions that can be expressed as infinite sums. In particular, functions of the form $\frac{1}{(1-x)^n}$ for positive integers n are crucial in various fields such as physics, engineering, and computer science. This article explores how to derive power series expansions for these functions, with a focus on $\frac{1}{(1-x)^2}$, $\frac{1}{(1-x)^3}$, and $\frac{1}{(1-x)^4}$, by leveraging the differentiation of the basic geometric series $\frac{1}{1-x}$.

Understanding the Geometric Series

Before diving into the expansions of the functions $\frac{1}{(1-x)^n}$, it is essential to understand the geometric series, which forms the foundation for these derivations.

Basic Geometric Series

The geometric series is one of the most well-known power series, expressed as:

$$
\frac{1}{1 - x} = \sum_{k=0}^\infty x^k,
$$

valid for $|x| < 1$. This series converges within the radius of convergence $|x| < 1$ and represents the sum of an infinite geometric progression with ratio x.

Key properties:
- The sum of the geometric series provides a simple power series expansion.
- Differentiating or integrating this series yields new series for related functions.

Deriving Power Series for $\frac{1}{(1-x)^n}$ by Differentiation

The main idea to find expansions for $\frac{1}{(1-x)^n}$ involves differentiating the basic geometric series with respect to x. This approach leverages the fact that differentiation of a power series term-by-term can produce series for functions involving higher powers in the denominator.

Step-by-step Approach

1. Start with the geometric series:

$$
\frac{1}{1 - x} = \sum_{k=0}^\infty x^k.
$$

2. Differentiate both sides with respect to x:

$$
\frac{d}{dx} \left(\frac{1}{1 - x} \right) = \frac{d}{dx} \left(\sum_{k=0}^\infty x^k \right).
$$

3. Compute derivatives:

$$
\frac{1}{(1 - x)^2} = \sum_{k=1}^\infty k x^{k-1}.
$$

4. Express as a power series centered at x:

$$
\frac{1}{(1 - x)^2} = \sum_{k=1}^\infty k x^{k-1}.
$$

5. Reindex the series for standard form:

Let $n = k - 1$, then:

$$
\frac{1}{(1 - x)^2} = \sum_{n=0}^\infty (n+1) x^n.
$$

This is the power series expansion for $\frac{1}{(1-x)^2}$ valid for $|x| < 1$.

Power Series for $\frac{1}{(1-x)^2}$

Using the differentiation method outlined, the power series expansion for $\frac{1}{(1-x)^2}$ is:

$$
\boxed{
\frac{1}{(1 - x)^2} = \sum_{n=0}^\infty (n + 1) x^n, \quad |x| < 1.
}
$$

Interpretation:
- The coefficients $(n+1)$ grow linearly, reflecting the derivative's influence.
- This series converges for all x in the interval $(-1, 1)$.

Deriving Power Series for $\frac{1}{(1-x)^3}$

The next step involves finding the power series expansion for $\frac{1}{(1-x)^3}$. We can obtain this by differentiating the series for $\frac{1}{(1-x)^2}$ with respect to x, or directly differentiating the geometric series twice.

Method 1: Differentiating $\frac{1}{(1-x)^2}$

1. Write the series for $\frac{1}{(1-x)^2}$:

$$
\frac{1}{(1-x)^2} = \sum_{n=0}^\infty (n+1) x^n.
$$

2. Differentiate both sides with respect to x:

$$
\frac{d}{dx} \left(\frac{1}{(1-x)^2} \right) = \frac{d}{dx} \left(\sum_{n=0}^\infty (n+1) x^n \right).

\]

3. Compute derivatives:

$$
\frac{2}{(1 - x)^3} = \sum_{n=1}^\infty n (n+1) x^{n-1}.
$$

4. Reindex the series:

Let $m = n - 1$:

$$
\frac{2}{(1 - x)^3} = \sum_{m=0}^\infty (m+1)(m+2) x^{m}.
$$

5. Divide both sides by 2:

$$
\frac{1}{(1 - x)^3} = \frac{1}{2} \sum_{m=0}^\infty (m+1)(m+2) x^{m}.
$$

Final Power Series:

$$
\boxed{
\frac{1}{(1 - x)^3} = \sum_{n=0}^\infty \frac{(n+1)(n+2)}{2} x^n, \quad |x| < 1.
}
$$

Power Series for $\frac{1}{(1-x)^4}$

Similarly, to find the expansion for $\frac{1}{(1-x)^4}$, we differentiate the series for $\frac{1}{(1-x)^3}$ or directly differentiate the geometric series three times.

Method: Differentiating $\frac{1}{(1-x)^3}$

1. Recall:

$$
\frac{1}{(1-x)^3} = \sum_{n=0}^\infty \frac{(n+1)(n+2)}{2} x^n.
$$

2. Differentiate both sides with respect to x:

$$
\frac{d}{dx} \left(\frac{1}{(1-x)^3} \right) = \frac{d}{dx} \left(\sum_{n=0}^\infty \frac{(n+1)(n+2)}{2} x^n \right).
$$

3. Compute derivatives:

$$
\frac{3!}{(1 - x)^4} = \sum_{n=1}^\infty n \frac{(n+1)(n+2)}{2} x^{n-1}.
$$

4. Recognize that $3! = 6$, so:

$$
\frac{6}{(1 - x)^4} = \sum_{n=1}^\infty n \frac{(n+1)(n+2)}{2} x^{n-1}.
$$

5. Reindex with $m = n - 1$:

$$
\frac{6}{(1 - x)^4} = \sum_{m=0}^\infty (m+1) \frac{(m+2)(m+3)}{2} x^{m}.
$$

6. Divide both sides by 6:

$$
\frac{1}{(1 - x)^4} = \frac{1}{12} \sum_{m=0}^\infty (m+1)(m+2)(m+3) x^{m}.
$$

Final Power Series:

$$
\boxed{
\frac{1}{(1 - x)^4} = \sum_{n=0}^\infty \frac{(n+1)(n+2)(n+3)}{12} x^n, \quad |x| < 1.
}
$$

Summary of Power Series Expansions

To consolidate, here are the derived power series expansions for the functions $\frac{1}{(1-x)^n}$:

- **For $n=2$:**

\[
\

Frequently Asked Questions

How can I find the power series expansion for 1/(1 - x)^2 using differentiation?

Start with the geometric series $1/(1 - x) = \sum_{n=0}^{\infty} x^n$ for $|x|<1$. Differentiating both sides with respect to x gives $d/dx[1/(1 - x)] = \sum_{n=1}^{\infty} n x^{n-1}$. Multiplying both sides by 1 to match the original function's form yields $1/(1 - x)^2 = \sum_{n=1}^{\infty} n x^{n-1}$.

What is the power series expansion for 1/(1 - x)^3 obtained by differentiating 1/(1 - x)^2?

Since $1/(1 - x)^2 = \sum_{n=1}^{\infty} n x^{n-1}$, differentiate this series with respect to x: $d/dx[1/(1 - x)^2] = \sum_{n=1}^{\infty} n(n - 1) x^{n-2}$. Recognizing that $d/dx[1/(1 - x)^2] = 2/(1 - x)^3$, we find $1/(1 - x)^3 = \sum_{n=2}^{\infty} n(n - 1) x^{n-2}$.

How can I derive the power series expansion for 1/(1 - x)^4 from the previous series?

Start with $1/(1 - x)^3 = \sum_{n=2}^{\infty} n(n - 1) x^{n-2}$ and differentiate it with respect to x: $d/dx[1/(1 - x)^3] = \sum_{n=2}^{\infty} n(n - 1)(n - 2) x^{n-3}$. Since $d/dx[1/(1 - x)^3] = 3!/(1 - x)^4 = 6/(1 - x)^4$, we get $1/(1 - x)^4 = \sum_{n=3}^{\infty} n(n - 1)(n - 2) x^{n-3}$.

What is the general approach to find power series expansions of (1/(1 - x))^k by differentiating 1/(1 - x)?

Begin with the geometric series $1/(1 - x) = \sum_{n=0}^{\infty} x^n$. Differentiate this series (k - 1) times with respect to x to generate higher powers of 1/(1 - x). Each differentiation introduces a factorial term and increases the power in the denominator, allowing you to express $(1/(1 - x))^k$ as a series involving factorial coefficients and powers of x.

Are there straightforward formulas for the coefficients in the power series expansions of (1/(1 - x))^n?

Yes, the coefficients in the expansion of (1/(1 - x))^n are given by the generalized binomial coefficients: C(n + k - 1, k) = (n + k - 1)! / (k! (n - 1)!), which appear in the series expansion ∑_{k=0}^∞ C(n + k - 1, k) x^k for |x|<1.

Additional Resources

Power Series Expansions: Unlocking the Infinite Potential of $\frac{1}{(1-x)^n}$

In the realm of mathematical analysis, power series serve as powerful tools that allow us to approximate and understand complex functions through infinite sums. Among these, the functions of the form $\frac{1}{(1-x)^n}$ are particularly significant, finding applications across calculus, differential equations, combinatorics, and mathematical modeling. Today, we delve into the fascinating process of deriving their power series expansions using a method as elegant as it is effective: differentiation of the geometric series.

This article offers an in-depth exploration of how to obtain these series expansions for $\frac{1}{(1-x)^2}$, $\frac{1}{(1-x)^3}$, and $\frac{1}{(1-x)^4}$ by leveraging the foundational series expansion of $\frac{1}{1-x}$. Whether you're a student, educator, or mathematical enthusiast, understanding these derivations enhances your conceptual toolkit and deepens your appreciation for the interconnectedness of mathematical concepts.

Understanding the Foundation: The Geometric Series

The Basic Series: $\frac{1}{1-x}$

The starting point for all our expansions is the geometric series:

$$
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{for } |x| < 1
$$

This fundamental series converges within the open interval $|x| < 1$, providing a bridge between simple algebraic expressions and infinite sums. Its simplicity makes it an ideal foundation for deriving more complex series via differentiation.

Why Differentiation? The Power of Derivatives in Series Expansion

Differentiation acts as a transformative tool that elevates the order of the functions involved, revealing new series expansions from existing ones. When differentiating a power series term-by-term, we gain access to derivatives of the original function, which correspond to related functions with higher powers in the denominator.

For example, differentiating $\frac{1}{1-x}$ with respect to x yields:

$$
\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{1}{(1-x)^2}
$$

This insight provides a straightforward pathway to obtaining the series expansion of $\frac{1}{(1-x)^2}$, which can then be extended further for higher powers.

Deriving Power Series for $\frac{1}{(1-x)^2}$

Step 1: Starting with the geometric series

Recall:

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]

valid for \(|x| < 1\).

Step 2: Differentiating both sides with respect to \(x\)

Differentiating term-by-term:

\[
\frac{d}{dx} \left(\frac{1}{1-x} \right) = \frac{d}{dx} \left(
\sum_{n=0}^{\infty} x^n \right)
\]

Calculating the derivatives:

\[
\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}
\]

Note that the sum now starts at \(n=1\) because the derivative of the constant term \(x^0=1\) is zero.

Step 3: Reindexing the series

To express the series in powers of \(x\), replace \(n-1\) with \(k\):

\[
k = n - 1 \Rightarrow n = k + 1
\]

Thus:

\[
\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k
\]

Final Series for \(\frac{1}{(1-x)^2}\):

\[
\boxed{

\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k, \quad |x| < 1
}
\]

Interpretation: Each coefficient \((k+1)\) reflects the linear growth of the sequence, with the series converging within the radius \(|x| < 1\).

Extending to \(\frac{1}{(1-x)^3}\) and \(\frac{1}{(1-x)^4}\)

The method used for \(\frac{1}{(1-x)^2}\) can be systematically extended to derive the series for higher powers by successive differentiation.

Power Series for \(\frac{1}{(1-x)^3}\)

Step 1: Differentiating \(\frac{1}{(1-x)^2}\)

From the previous result:

\[
\frac{1}{(1-x)^2} = \sum_{k=0}^{\infty} (k+1) x^k
\]

Differentiate both sides with respect to \(x\):

\[
\frac{d}{dx} \left(\frac{1}{(1-x)^2} \right) = \frac{d}{dx} \left(
\sum_{k=0}^\infty (k+1) x^k \right)
\]

Calculating derivatives:

\[
\frac{2}{(1-x)^3} = \sum_{k=1}^\infty k (k+1) x^{k-1}
\]

Note: The sum on the right starts at \(k=1\) because the derivative of the \(k=0\) term is zero.

Step 2: Reindexing to powers of \(x\)

Let \(m = k - 1\):

\[
k = m + 1
\]

Thus:

\[
\frac{2}{(1-x)^3} = \sum_{m=0}^\infty (m+1)(m+2) x^{m}
\]

Divide both sides by 2:

\[
\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{m=0}^\infty (m+1)(m+2) x^{m}
\]

Final Series for \(\frac{1}{(1-x)^3}\):

\[
\boxed{
\frac{1}{(1-x)^3} = \sum_{m=0}^\infty \frac{(m+1)(m+2)}{2} x^{m}, \quad
|x| < 1
}
\]

Power Series for \(\frac{1}{(1-x)^4}\)

Step 1: Differentiating \(\frac{1}{(1-x)^3}\)

From the previous result:

\[
\frac{1}{(1-x)^3} = \sum_{m=0}^\infty \frac{(m+1)(m+2)}{2} x^{m}
\]

Differentiate both sides:

\[
\frac{d}{dx} \left(\frac{1}{(1-x)^3} \right) = \frac{d}{dx} \left(

\sum_{m=0}^\infty \frac{(m+1)(m+2)}{2} x^{m} \right)
\]

Left side derivative:

\[
\frac{3!}{(1-x)^4} = \frac{6}{(1-x)^4}
\]

Right side derivative:

\[
\sum_{m=1}^\infty m \frac{(m+1)(m+2)}{2} x^{m-1}
\]

Note: The sum starts from \(m=1\).

Step 2: Reindexing to powers of \(x\)

Let \(p = m - 1\):

\[
m = p + 1
\]

So:

\[
\frac{6}{(1-x)^4} = \sum_{p=0}^\infty (p+1) \frac{(p+2)(p+3)}{2} x^{p}
\]

Divide both sides by 6:

\[
\frac{1}{(1-x)^4} = \frac{1}{12} \sum_{p=0}^\infty (p+1)(p+2)(p+3) x^{p}
\]

Final Series for \(\frac{1}{(1-x)^4}\):

\[
\boxed{
\frac{1}{(1-x)^4} = \sum_{p=0}^\infty \frac{(p+1)(p+2)(p+3)}{12} x^{p},
\quad |x| < 1
}
\]

Summary of Power Series Expansions

| Function | Power Series

Find A Power Series Expansions For 1 1 X 2 1 1 X 3 1 1 X 4 By Differentiating 1 1 X Give

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